Emerging Scholars Program – Fall 2007

M210E – Calculus Workshop

Problem Set 9

"An expert problem solver must be endowed with two incompatible qualities, a restless imagination and a patient pertinacity." – Howard W. Eves

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49. A path to power. Find a power series representation for each of the following functions, centered at $x = 0$. For extra practice, find a power series representation for each function, this time centered at $x = 1$.

$$\frac{1}{2-x} \qquad \frac{1}{(2-x)^2} \qquad \frac{x}{(2-x)^2}$$

50. Some Taylor-made functions. Write a Taylor series, centered at $x = 0$, for each of the following functions.

$$e^x \qquad e^{-x^2} \qquad e^{1-x^2} \qquad \int_0^x e^{1-t^2}\,dt$$

51. More manufactured power series. Find a power series for each of the following functions, centered at $x = 0$. (Keep in mind that a Taylor series is a specific kind of power series.)

$$x\cos(x^2) \qquad \log(3+x) \qquad \frac{1}{x^2-2x-3} \qquad \sin^2 x$$

52. From the general to the specific. Use what you know about power series to evaluate the sum

$$\frac{1}{1\cdot 2^1}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}+\cdots$$

53. Generatingfunctionology. Recall that the Fibonacci series $\{F_n\}_{n=0}^{\infty}$, defined in Problem 22, is defined by $F_0 = F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Define a function $f(x)$ by

$$f(x) = \sum_{n=0}^{\infty} F_n x^n.$$

1. Prove that $f(x) = 1 + xf(x) + x^2 f(x)$. Use this information to give a formula for $f(x)$ that does not require the use of power series.
2. Use this formula to obtain a power series representation for $f(x)$, centered at $x = 0$, with explicitly defined coefficients (rather than the coefficients F_n, which are recursively defined). (*Hint:* In this problem, you'll encounter a quadratic polynomial that cannot be factored over the integers. Still, you may find it useful to break it up into factors of the form $(1-\alpha x)(1-\beta x)$.)
3. Given that a function can have only one power series centered at a given value of x, what can you conclude about the numbers F_n?

If you complete this problem, you will arrive at a lovely formula called *Binet's Fibonacci number formula.* This problem is an example of the method of *generating functions*, which is used extensively in combinatorics. If you'd like to learn more about the subject, you can download Herbert Wilf's book *generatingfunctionology* from his webpage at the University of Pennsylvania.

54. Ingenuity: Do I know you? One hundred mathematicians show up for an afternoon tea at a conference. Some of the mathematicians know each other, but some do not. Given that "knowing" is a symmetric relationship – that is, if A knows B, then B knows A – prove that there must be two mathematicians who know the same number of people at the tea.