Emerging Scholars Program – Fall 2007
M210E – Calculus Workshop
Problem Set 12

"Whoever . . . proves his point and demonstrates the prime truth geometrically should be believed by all the world, for there we are captured." – Albrecht Dürer

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67. Vector products: piffle or not piffle? For each of the following "products" of vectors in $\mathbb{R}^3$, determine whether the given expression makes sense. (Assume that each dot is a dot product, and each cross is a cross product; that is, neither of these symbols represents real number multiplication or scalar multiplication in this problem.) If an expression does make sense, specify whether the value of the expression is a scalar or a vector.

1. $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$
2. $(\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$
3. $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
4. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

68. Algebra and the cross product. Prove or disprove each of the following statements:

1. For all $\mathbf{a}$, $\mathbf{b}$ in $\mathbb{R}^3$ and all $\alpha \in \mathbb{R}$, $(\alpha\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\alpha\mathbf{b}) = \alpha(\mathbf{a} \times \mathbf{b})$.
2. For all $\mathbf{a}$, $\mathbf{b}$ in $\mathbb{R}^3$, $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}$.
3. For all $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ in $\mathbb{R}^3$, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.
4. For all $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ in $\mathbb{R}^3$, $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$.

69. The many faces of the cross product. Use the cross product (along with the other techniques at your disposal) to answer the following questions.

1. Find all vectors that all perpendicular to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} - \mathbf{k}$.
2. Suppose that $\mathbf{v}$ and $\mathbf{w}$ are vectors having length 3 and 4, respectively. If $\mathbf{v} \times \mathbf{w}$ has length 5, find all the possible values of the measure of the angle between $\mathbf{v}$ and $\mathbf{w}$.
3. Find a nonzero vector that lies in the plane whose equation is $x + y + 3z = 0$ and the plane whose equation is $2x - 5y = 0$.
4. Find the area of the parallelogram in $\mathbb{R}^3$ defined by the vectors $\langle 1, -1, 1\rangle$ and $\langle 2, -1, -1\rangle$.
5. Find the area of the triangle in $\mathbb{R}^2$ whose vertices are $(14, 92)$, $(17, 76)$, and $(18, 12)$.
6. Find all values of c such that the vectors $\langle 3, 2, 1\rangle$, $\langle -4, c, 1\rangle$, and $\langle 0, 3, c\rangle$ are coplanar. (Note that having three vectors coplanar is different from having three *points* coplanar. It is a geometric fact that any three points are coplanar.)

70. The dot products of the many faces. One task that your beloved AI often has to perform in his research is the construction of polyhedra in $\mathbb{R}^3$ (or in the 3-sphere or in hyperbolic 3-space, depending on the situation) whose faces meet at specified angles. In this problem and the next, we'll develop a sense of what this task entails. Consider the bounded polyhedron whose faces are given by the equations

$$\begin{aligned} x &= y \\ y &= z \\ z &= 0 \\ x+y &= 1 \end{aligned}$$

First, sketch this polyhedron. (You may find it helpful to try to find the vertices of the polyhedron. Note that, in this case, each vertex is the intersection point of three faces.) Then for each pair of faces, find the angle between the two faces. Record your findings in a table. (Note that finding the angle between two faces of a polyhedron is the same as finding the angle between two planes. Also, recall that you can find the angle between two planes by finding the angle between their normal vectors.)

71. Now it's your turn. Find a polyhedron in $\mathbb{R}^3$ with faces A, B, C, D, such that

1. Faces A and B meet at a 45° angle.
2. Faces B and C meet at a 60° angle.
3. Faces C and D meet at a 45° angle.
4. Each pair of faces not specifically mentioned above meets at a 90° angle.

72. Ingenuity: A flurry of factorials. Suppose we find the value of the expression shown below. If we were to write down the value of this sum, what would be the rightmost three digits?

$$0! + 1! + 2! + 3! + 4! + \cdots + 999! + 1000!$$