Emerging Scholars Program – Fall 2007

M210E – Calculus Workshop

Problem Set 13

"Let no man ignorant of geometry enter here." – Plato

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73. Covering all the angles. When we talk about geometry in $\mathbb{R}^3$, there are several different situations in which we are interested in finding angles. We can find angles between two lines, angles between a line and a plane, and angles between two planes. Furthermore, there are several different ways to specify a plane: one way is to specify two vectors that lie in the plane; another way is to give the equation of the plane. Similarly, a line can be specified in several different ways. Each situation requires us to use a slightly different technique to compute the desired angle. To help you hone your intuition for which situation requires which technique, try computing the following angles. You'll probably need a calculator to find the degree measures of most of the angles.

1. The angle between the line through the origin containing the vector $\langle 1, 1, 0\rangle$ and the line through the origin containing the vector $\langle 0, -1, 1\rangle$.
2. The angle between the line $\{(3, 1 + 2t, -t) : t \in \mathbb{R}\}$ and the line $\{(u, 3 - 2u, -1 + u) : u \in \mathbb{R}\}$. (*Caution:* Keep in mind that it is possible for two lines not to intersect!)
3. The angle between the line through the origin containing the vector $\langle 4, -2, 1\rangle$ and the plane through the origin containing the vectors $\langle 2, 0, 2\rangle$ and $\langle 0, 3, -4\rangle$.
4. The angle between the line $x = y + 1 = 3z - 2$ and the plane $2x + y = 4z - 1$.
5. The angle between the plane containing the points $(1, 1, 2)$, $(0, 4, 5)$, and $(-3, 0, -1)$, and the plane containing the points $(3, 0, 0)$, $(-1, -2, 3)$, and $(5, 1, -4)$.

74. Pump up the volume? Let a, b, and c be real numbers. Find, in terms of a, b, and c, the volume of the parallelepiped which has a vertex at the origin and has the vectors $\langle 1, 0, 0\rangle$, $\langle a, 1, 0\rangle$, and $\langle b, c, 1\rangle$ as edges. What happens to the volume of the solid as a, b, and c increase?

75. Planes and projections. Suppose that $\mathbf{v}$ is a vector in $\mathbb{R}^3$, and $\mathcal{P}$ is a plane through the origin. The *orthogonal projection of* $\mathbf{v}$ *onto* $\mathcal{P}$ is defined to be a vector $\mathbf{w}$ with the following properties:

1. The vector $\mathbf{w}$ is in the plane $\mathcal{P}$.
2. The vector $\mathbf{v} - \mathbf{w}$ is orthogonal to the plane $\mathcal{P}$.

Suppose we are given two vectors $\mathbf{a}$ and $\mathbf{b}$ that lie in the plane $\mathcal{P}$. Give a formula, in terms of $\mathbf{v}$, $\mathbf{a}$, and $\mathbf{b}$, for the orthogonal projection of $\mathbf{v}$ onto $\mathcal{P}$. (You may use any vector operations you wish, such as norms, dot products, and cross products. Also, if you wish, you may use the coordinates of a given vector; that is, if $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, you may have the variables v_1, v_2, and v_3 in your formula.)

76. Into the fourth dimension. Let $\mathbb{R}^4$ denote four-dimensional coordinate space; that is, the set of all points (w, x, y, z), where w, x, y, and z are real numbers. Just as three generic points determine a plane in $\mathbb{R}^3$[1], three generic points in $\mathbb{R}^4$ determine a plane in $\mathbb{R}^4$. Let $\mathcal{P}$ be the plane in $\mathbb{R}^4$ containing the points $(1, 1, 0, 0)$, $(2, 0, 0, 1)$, and $(0, 1, 2, 1)$, and let $\mathcal{Q}$ be the plane in $\mathbb{R}^4$ containing the points $(1, 0, -1, 0)$, $(2, 2, 0, 1)$, and $(-1, 1, -1, 1)$. Describe the intersection of the planes $\mathcal{P}$ and $\mathcal{Q}$; that is, the set of points that lie on both planes.

77. Determinants, diagonals, and the fourth dementia. One way to determine whether four vectors in $\mathbb{R}^4$ are in the same hyperplane – that is, a three-dimensional subspace of $\mathbb{R}^4$, similar to a plane in $\mathbb{R}^3$ – is to compute the determinant of the matrix whose columns are the four vectors. Suppose, for example, we want to determine whether $\langle 2, 0, 1, 3\rangle$, $\langle 0, -1, -1, 0\rangle$, $\langle 2, -2, 0, 4\rangle$, and $\langle -1, -1, 3, 3\rangle$ lie in a common hyperplane. It is a theorem from linear algebra that these vectors lie in the same hyperplane if and only if the determinant

$$\begin{vmatrix} 2 & 0 & 2 & -1 \\ 0 & -1 & 0 & -1 \\ 1 & -1 & 0 & 3 \\ 3 & 0 & 4 & 3 \end{vmatrix}$$

is equal to zero. Use expansion by minors to calculate this determinant. Once you have done this, try computing this determinant by multiplying along the diagonals, and see if you get the same answer.

78. Ingenuity: A Platonic relationship. A *Platonic solid* (or *regular solid*) is a polyhedron whose faces are all regular polygons and are all congruent to each other, and which has the same number of edges meeting at each vertex. *Euler's formula* states that *every* polyhedron, whether Platonic or not, satisfies the condition

$$V - E + F = 2,$$

where V is the number of vertices of the polyhedron, E is the number of edges, and F is the number of faces. Use this theorem to prove that there are only finitely many Platonic solids. Can you find a complete list?

[1]Here, the word "generic" means that there are some combinations of three points that do not determine a unique plane, as happens when the three points are collinear. However, in a probabilistic or measure-theoretic sense, "most" combinations of three points do determine a unique plane.