Emerging Scholars Program – Fall 2007
M210E – Calculus Workshop
Problem Set 14

"Time was when all the parts of the subject were dissevered, when algebra, geometry, and arithmetic either lived apart or kept up cold relations of acquaintance confined to occasional calls upon one another; but that is now at an end; they are drawn together and are constantly becoming more and more intimately related and connected by a thousand fresh ties, and we may confidently look forward to a time when they shall form but one body with one soul." – J. J. Sylvester

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79. Equations on demand. Give an equation for each of the following:

1. The plane passing through the origin and perpendicular to the vector $\langle 0, 1, 1\rangle$.
2. The plane passing through the point $(2, -3, -3)$ and perpendicular to the vector $\langle -1, 0, -4\rangle$.
3. The plane passing through the points $(1, 4, 2)$, $(-2, 0, -1)$, and $(0, 3, -3)$.
4. The plane passing through the line $2x - 1 = y + 3 = 4z$ and the point $(1, 1, 2)$.
5. The line through the points $(2, 1, 2)$ and $(-1, -3, -5)$.
6. The line through the point $(-1, -1, 2)$ perpendicular to the plane $x - z = 3$.
7. The line through the point $(5, 2, 0)$ parallel to the planes $4x - y - z = 1$ and $x + 2y + 2z = 3$.

80. Given two lines, find a plane. Show that the two lines given by the equations

$$\frac{x-1}{2} = -y + 4 = \frac{z}{3} \qquad \text{and} \qquad x = y = z$$

lie in the same plane. Then, find an equation in standard form for this plane.

81. Given two planes, find a line. Show that the two planes given by the equations

$$3x - 2y - z = 4 \qquad \text{and} \qquad 2x + y + 3z = 1$$

intersect in a line. Then, find a symmetric equation for this line.

82. Hey, this stuff is useful in real life (competitions). In his spare time, your beloved AI coaches a local high school math team. They just took the first round of the Mandelbrot Competition, a nationally respected high school math contest. The last problem on this month's individual test was the following: "Let $ABCD$ be a regular tetrahedron with edges of length 4. A plane passes through the edges AC, AD, BC, and BD at points W, X, Y, and Z, respectively. If $AW = 2$, $AX = 3$, and $BZ = 1$, find the distance BY." There are several ways to solve this problem, such as an approach that uses Menelaus' Theorem, an obscure theorem from plane geometry. However, one can also solve this problem using the techniques of analytic geometry that you have learned over the last two weeks. Give it a try. (*Hint:* You may find it useful to use the fact that the tetrahedron in $\mathbb{R}^3$ with vertices at the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(1, 1, 1)$ is a regular tetrahedron, even though its edges do not have length 4. The essential information in this problem is contained in the ratios in which the edges are cut; for example, the fact that the plane bisects the edge AC.)

83. Polyhedra . . . in 4D! We can define a *four-dimensional hypercube* to be the convex hull of (that is, the polyhedron formed by) the vertices $(\pm 1, \pm 1, \pm 1, \pm 1)$. Note that each coordinate may be chosen independently of the others; therefore, there are sixteen vertices. (To help you see why this is a logical way to produce a cube-like object, think about the polyhedron in $\mathbb{R}^3$ whose vertices are the points $(\pm 1, \pm 1, \pm 1)$.)

1. We have already seen that this hypercube has 16 vertices. How many edges does it have? How many two-dimensional faces? How many three-dimensional faces?
2. Suppose that, instead of a four-dimensional hypercube, we construct a hypercube in n dimensions. Give a formula, in terms of n and k, for the number of k-dimensional faces this hypercube will have.

84. Ingenuity: The Chicken McNuggets Theorem. The *Chicken McNuggets Theorem* is an elegant theorem in elementary number theory that states that, if Chicken McNuggets are available only in boxes of m or n nuggets each, then the greatest number of nuggets that we cannot obtain exactly is $mn - m - n$, *provided that m and n do not have any common prime factors.* Note that this theorem actually makes two assertions: that we cannot obtain exactly $mn - m - n$ nuggets, *and* that we can obtain any number of nuggets greater than $mn - m - n$. Following are some questions about this theorem and its applications.

1. There was a time, when I was a kid (back in the 1980's), when Chicken McNuggets were available in boxes of 6 or 9. Explain why we cannot use the Chicken McNuggets Theorem in this situation. What is the greatest number of nuggets that we cannot obtain exactly?
2. Of course, the Chicken McNuggets Theorem applies to disciplines other than fast food, such as football. Suppose that, during a particular football game, the Texas Longhorns only score touchdowns (worth 7 points each, since they always make the ensuing point-after attempt), and field goals (worth 3 points each). What is the greatest score that the Longhorns cannot obtain?
3. Back to the fine dining establishment previously referenced: as I got older (around 1990 or so), the fast food chain started selling Chicken McNuggets in boxes of 4 and 20, in addition to the pre-existing boxes of 6 and 9. In this situation, what was the largest number of nuggets one could not obtain exactly? (It may be difficult to apply the theorem directly here. Try reasoning your way through this one.)
4. In the year 3500, the aforementioned fast food chain will have grown into an interplanetary corporation, with locations all over the Milky Way galaxy. At this time, they will sell their extremely hard, yet mysteriously edible Meteorite McNuggets in boxes of 6 and 9, as well as in the Intergalactic Army Value Pack, which will contain two million nuggets. In this situation, what will be the largest number of nuggets that cannot be obtained exactly?