Emerging Scholars Program – Fall 2007
M210E – Calculus Workshop
Problem Set 15

"There is a tradition of opposition between adherents of induction and of deduction. In my view it would be just as sensible for the two ends of a worm to quarrel." – Alfred North Whitehead

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85. Trace it out. Graph each of the following vector-valued functions in $\mathbb{R}^2$ or $\mathbb{R}^3$, whichever is appropriate.

1. $\mathbf{f}(t) = (3t - 1)\mathbf{i} + (2t + 3)\mathbf{j}$
2. $\mathbf{f}(t) = (t - 2)\mathbf{i} + t^2\mathbf{j}$
3. $\mathbf{f}(t) = (\sin t)\mathbf{i} + t\mathbf{j}$
4. $\mathbf{f}(t) = (\cos^2 t)\mathbf{i} + (\sin^2 t)\mathbf{j}$
5. $\mathbf{f}(t) = e^t\mathbf{i} + e^t\mathbf{j} + e^{2t}\mathbf{k}$
6. $\mathbf{f}(t) = t\mathbf{i} + (2\cos t)\mathbf{j} + (\sin t)\mathbf{k}$

86. Having a ball. Despite constant warnings from Cody that what he is doing "is extremely dangerous" and "may cost him his grade in M210E," Jackson swings a one-kilogram ball, attached to a one-meter rope, in a circle around his head, keeping his hand stationary over his head. (He says it's a "physics experiment," though the rest of his group knows it's about as much of a "physics experiment" as his "tornado of iron filings.") Using the laws of Newtonian mechanics that you learned in high school or in your physics class this semester, answer the following questions:

1. Suppose the ball is revolving around Jackson's head at a rate of k revolutions per second, at a height of h meters; and the ball remains a constant r meters from the center of Jackson's head. Model the path of the ball as a vector-valued function; that is, write a vector-valued function in $\mathbb{R}^3$ that represents the position of the center of the ball at time t, with the place on the ground where Jackson is standing as the origin. (Your answer, of course, will be in terms of k, h, and r.)
2. Find the velocity, speed, and acceleration of the ball at time t. (Recall that velocity is given as a vector, while speed is a number. Acceleration, like velocity, is given as a vector unless otherwise specified.)
3. At time t, what force is Jackson applying to the ball to keep it in its potentially deadly "orbit"? (Force is also given as a vector. Be sure to take gravity into account.)
4. From time $t = 0$ to time $t = t_0$, how much work does Jackson do on the ball? (*Hint:* The answer to this question underscores one of Cody's numerous objections to Jackson's current activity.)
5. Assuming that Jackson swings the ball so that it remains a constant height of 1.73 meters over the ground, and he holds the other end of the rope at a height of 2.19 meters over the ground, how many revolutions per second will the ball make if Jackson applies a force just large enough to keep the ball in orbit? (You'll probably need a calculator for this part.)

87. This is getting ridiculous. To Cody's complete chagrin, Jackson's fellow UTeach-er William is so intrigued by the "orbiting ball of doom" that he decides to give it a try. William extends his swinging hand away from his body and swings the ball in a vertical circle so that the ball just grazes the ground at the lowest point of its orbit. (In the meanwhile, Cody sits in a safe corner of the room, visions of personal injury lawyers dancing in his head.) Assuming the ball moves at a constant k revolutions per second, how much work does William do on the ball between time $t = 0$ and time $t = t_0$?

88. Lines and spirals. Consider the spiral in $\mathbb{R}^3$ given by the vector-valued function

$$\mathbf{f}(t) = \langle \sin t, \cos t, t \rangle.$$

If $t_0 = \sin^{-1}(\frac{3}{5})$, what is the vector $\mathbf{f}(t_0)$? Give an equation, in symmetric form, for the line tangent to the spiral at this point.

89. For the geometrically "inclined." At what angle does the spiral described in Problem 88 pass through the xy-plane? At what angle does the spiral pass through the plane $z = 2007$?

90. Ingenuity: Ramsey theory meets international diplomacy. Six countries send representatives to an international conference. For any two countries A and B, we can say that either A and B are allies, or A and B are enemies.

1. Prove that there must be three countries represented at the meeting that are either mutual allies, or mutual enemies.
2. Prove or disprove that the previous statement holds if there are only five countries represented at the meeting.
3. Suppose that we make the additional assumption that "an enemy of an enemy is a friend"; that is, if A and B are both enemies of C, then A and B are allies. How many countries must be represented at the meeting in order to guarantee that there are four countries that are mutual allies?