

Emerging Scholars Program – Fall 2007  
M210E – Calculus Workshop  
Problem Set 19

*“The modern, and to my mind true, theory is that mathematics is the abstract form of the natural sciences; and that it is valuable as a training of the reasoning powers not because it is abstract, but because it is a representation of actual things.” – T. H. Sanford*



- 109. Some exercises to get the ball rolling.** Define  $f(x, y) = x^2 + y^2 + 4y$ . Answer as many of the following questions as you can:
1. What are the first and second partial derivatives of  $f$ ?
  2. Does  $f$  have any local extrema? If so, where are they?
  3. What is the gradient of  $f$  at the point  $(x, y)$ ? Sketch the gradient field.
  4. What is the directional derivative of  $f$  at the point  $(1, 2)$  in the direction  $\langle 12, -5 \rangle$ ?
  5. Consider the surface given by the equation  $z = f(x, y)$ . If you placed a ball at the point  $(1, 1, 6)$  on this surface, in what direction would it begin to roll? How fast would it accelerate (assuming the physical conditions are those on the planet Earth)?
- 110. Return to normalcy.** Use the gradient to find a normal vector to the surface given by the equation  $x^2 + y^2 - z^2 = 1$  at the point  $(7, 4, 8)$ . Then, use this information to find the tangent plane to the surface at this point. (You should be able to do this without ever writing  $z$  as a function of  $x$  and  $y$ .)
- 111. Team Newton, at a crossroads.** Jo, Michael, and Alya are at an intersection of two roads in a particularly hilly part of Austin. Jo turns thirty degrees east of north and looks up one of the roads; she notices a sign that says that the road is at a 10% incline (going up). Michael turns forty-five degrees west of north and looks down the other road; he notices a sign that says that the road is at a 5% incline (going down). Assuming that the surface of the land around this intersection is smooth (*i.e.* differentiable), if Alya were to look due south from the same point, what incline would she see? (Feel free to use a calculator on this problem.)
- 112. Keep your distance.** What point on the ellipse given by the equation  $4x^2 + 9y^2 = 36$  is farthest from the point  $(5, 5)$ ? What point on the ellipse is closest to the line  $x + 7y = 35$ ?
- 113. FunctionQuest III: Saddle up.** Give an example of a function  $f(x, y)$  with the following properties, or prove that no such function exists:
1. All the derivatives of  $f$  exist and are continuous.
  2. The first partials  $f_x(0, 0)$  and  $f_y(0, 0)$  are equal to zero.
  3. The second partials  $f_{xx}(0, 0)$  and  $f_{yy}(0, 0)$  are positive.
  4. The function  $f$  does not have a local minimum at  $(0, 0)$ .

**114. Ingenuity: Not a position statement. Not at all.** Suppose we take Barry Bonds' home run record-breaking baseball and divide it into at least five regions by drawing arcs on the surface of the ball in such a way that, at any point where an arc terminates (we'll call such a point a "vertex"), there are exactly three arcs that terminate at that point. (Ignore the stitching and other MLB-related markings that are already on the ball.) You then play the following game. First, you draw an asterisk on one region of the ball. Then Barry Bonds signs his name on another region of the ball. The game continues, with the two players taking turns putting asterisks/autographs on previously-unmarked regions of the ball. The winner is the player who first succeeds in marking three regions that share a common vertex. If you and Bonds both use the best possible strategy, will you win the game, will Bonds win the game, or will the game end in a tie (with all regions marked but neither player having marked all three faces around a single vertex)? (This problem is from the 2002 Putnam Competition; the original version of the problem used a polyhedron with "faces" and "edges" in place of a ball with "regions" and "arcs", but this does not make any difference. If you enjoy problems like this, you may want to consider taking this year's Putnam exam, though I am not sure whether it is too late to sign up. I can tell you whom to talk to if you are interested.)