Emerging Scholars Program – Fall 2007
M210E – Calculus Workshop
Problem Set 20

"Inequality is the cause of all local movements." – *Leonardo da Vinci*

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115. The closest point. Find the point on the ellipse $x^2 + 4y^2 = 4$ that is closest to the line $5x + 3y = 15$. (Super bonus points if you can find two different ways to solve this problem. Super-duper bonus points if you can find three different ways.)

116. Big D. Find the maximum and minimum values of the function

$$f(x, y) = xye^{x+y}$$

on the D-shaped region defined by the inequalities $x \geq -2$ and $x + y^2 \leq 1$. (You may need to use a calculator on this problem; I didn't pay particular attention to making the numbers work out well.)

117. Squared segments. A point Q is placed in the triangular region whose vertices are $A = (0, 0)$, $B = (2, 0)$, and $C = (0, 3)$. Given that Q may be either in the interior of the region or on an edge or vertex, find the greatest and smallest possible values of the expression $AQ^2 + BQ^2 + CQ^2$. (Here, the symbols AQ, BQ, and CQ denote distances.)

118. It's really quite simple(x). Let $f(x, y)$ be a linear function of two variables; that is, a function of the form

$$f(x, y) = ax + by + c,$$

where a, b, and c are real numbers. Let P be a polygonal region in $\mathbb{R}^2$. Suppose we want to find the maximum value of f on the region P. The fact that this maximum value exists is a theorem of calculus, which should have been mentioned (but probably not proven) in class. Given the fact that f attains a maximum value on the region P, prove that this maximum value must be attained at a vertex of P. (It is possible that the maximum will also occur at a point in the interior of P, for example, but it is guaranteed to occur at at least one vertex. This theorem is the basis for an algorithm in linear programming called the *simplex method*, hence the cheesy name of this problem.)

119. The AM-GM Inequality. Suppose that $x_1, x_2, \ldots, x_n$ are nonnegative real numbers such that

$$x_1 + x_2 + \cdots + x_n = C,$$

where C is a given (fixed) real number. In terms of C, what is the greatest possible value of the product $x_1 x_2 \cdots x_n$? (Note that we are allowed to move the x_i's around, subject to the constraint that their sum is equal to C.) Show that your result implies the *AM-GM Inequality*, which states that for nonnegative real numbers $x_1, x_2, \ldots, x_n$, we have

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

Under what condition(s) does equality hold? (And where does the name "AM-GM" come from?)

120. Sets of SETs. The popular card game SET is played with a deck of eighty-one cards. Each card carries a different combination of the following four traits:

- **Shape:** The card may contain ovals (O), diamonds (D), or rectangles (R).
- **Color:** The shape(s) on the card may be colored red (R), green (G), or blue (B) (note that all shapes on a card have the same color).
- **Shading:** The shape(s) on the card may be filled in with the given color (F), partially filled in (P), or not filled in (E) (in which case the outlines of the shapes are still in the given color).
- **Number:** The card may have one (1), two (2), or three (3) of the given shape.

Note that there are $3 \cdot 3 \cdot 3 \cdot 3 = 81$ possible combinations of the four traits; therefore, each possible combination is carried by a single card. The goal of the game SET is to identify as many SETs of three cards as possible. A combination of three cards is a SET if, for each of the four traits (shape, color, shading, number), all three cards are the same in that trait, or all three cards are different in that trait. So for example:

OBF2, DRF2, and RGF2 form a set.
RRE3, DRE1, and ORP2 do not form a set.
DBE1, OGF3, and RRP2 form a set.
OBE3, DBF3, and DRP3 do not form a set.

Answer the following questions:

1. How many different three-card SETs are there in the deck? (Note that this will include SETs that overlap each other; for example, the card OBF2 may appear in several different SETs.)
2. Suppose that seventy-eight of the cards in the deck have been divided into groups of three so that they form twenty-six SETs. Do the remaining three cards form a SET?