Emerging Scholars Program – Fall 2007
M210E – Calculus Workshop
Problem Set 24

"And the kids, they love the Jell-O." – Austin Gatlin, channeling Bill Cosby

Instead of giving you another cleverly chosen math quote, I thought I would use this space to thank you all for giving me the opportunity to teach you this semester. I can't tell you how lucky I consider myself – most graduate students in my department, when they go to class, feel like they are going to work. When I go to class, I feel like I'm spending three hours having fun with a bunch of good friends. And that is the wish that I have for each of you – that in the next four years of your life, you will discover a career that not only allows you to nourish your body, but also nourishes your mind and your soul. Don't settle for anything less, because you deserve the best. – Cody L. Patterson, 06 December 2007

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139. Ingenuity: The small-number paradox. We will say that a positive integer is "small" if it can be described precisely, in English, in one hundred words or less. (Using hyphens and other dirty tricks to extend words is cheating; for purposes of this exercise, the phrase "forty-two" is two words, not one.) For example, the number 2 can be described as "two" (one word), "the integer after one" (four words), or "the only even prime number" (five words), to give a few examples. The number $142,857$ is a little trickier, but can still be described well within the one-hundred-word threshold:

One hundred forty two thousand eight hundred fifty seven (nine words)
The closest integer to one seventh of one million (nine words)

Meredith and André are having an argument (as is their wont), this time over the question of whether every positive integer is small. They reason as follows:

Meredith: "There are only finitely many words in the English language; let's say there are N. Then only N numbers can possibly be described using one word; only N^2 numbers can be described using two words, and so on. Even if you consider the fact that a description can have any length up to a hundred words, and allow for nonsense descriptions like 'turkey justice alliteration squad,'[1] there can only be $N + N^2 + N^3 + \cdots + N^{100}$ small numbers. But there are infinitely many positive integers, so not all of them can be small."

André: "Well, suppose that there *are* numbers that aren't small. Then one of them has to be the smallest, since any non-empty collection of positive integers has a smallest element. So take the smallest non-small number, and describe it as 'The smallest positive integer that is not small.' That's a contradiction, so there can't be positive integers that aren't small."

Who is right?

140. Ingenuity: A dice-y problem. How many different ways are there to label the sides of a cubical die with the numbers one through six? (Assume that two dice are the same if one can be rotated so that it is identical to the other.) How many different ways are there to do the labelling if we assume (as is the case with standard dice) that the numbers on opposite sides must add up to seven?

[1] Or are such things really nonsense? It turns out that the sentence "Buffalo buffalo Buffalo buffalo buffalo buffalo Buffalo buffalo." is in fact a grammatically correct sentence in the English language. And you thought the "Malkovich Malkovich?" scene in *Being John Malkovich* was a linguist's nightmare.

141. Ingenuity: Here's your sine. Suppose that ΔABC is a triangle with sides of length a, b, and c (with the side of length (lower-case letter) being opposite the vertex (same letter in upper-case)). Circumscribe a circle around ΔABC, and let R be the radius of this circle. Prove that

$$2R = \frac{a}{\sin \angle A}.$$

Explain how this theorem proves the Law of Sines.

142. Ingenuity: A self-referential number. Find a ten-digit number having the following properties: The first digit (from the left) is the number of zeroes in the number, the second digit is the number of ones in the number, and so on, down to the last (rightmost) digit, which is the number of nines in the number.

143. Ingenuity: Chomp! In the game of *Chomp*, two players take turns eating squares from a grid. The game board starts out as a rectangular grid (the dimensions of the rectangle can be chosen freely, with each choice of dimensions leading to a slightly different game). On each turn, a player must eat at least one square; and when a player eats a square, he/she must also eat all of the squares below and to the left of it. (Let your friendly AI and/or SA know if you want to work on this problem, so that they can show you what is meant by this – it makes more sense when illustrated with an example.) The upper-right corner square is designated as "poison"; the player who is forced to eat this square loses the game. Suppose the initial game board is an n-by-n square. If both players play as well as possible, who wins the game?

144. Ingenuity: Fifty-one's a crowd. Suppose we choose fifty-one integers from the set $\{1, 2, 3, 4, \ldots, 100\}$.

1. Prove that there must be two integers among the fifty-one we chose that are relatively prime (that is, do not have a common prime factor).
2. Prove that there must be two integers among the fifty-one we chose such that one of the integers is a multiple of the other.