We have

$$0^2 - 2 = -2 \equiv 2 \pmod 4$$
(1) $$1^2 - 2 = -1 \equiv 3 \pmod 4$$
$$2^2 - 2 = 2 \equiv 2 \pmod 4$$
$$3^2 - 2 = 7 \equiv 3 \pmod 4$$

Now we know that every integer x is congruent to
(1) exactly one of the integers $0, 1, 2, 3$ mod 4;
and if $x \equiv r \pmod 4$, then $x^2 - 2 \equiv r^2 - 2 \pmod 4$,
(1) by theorem 4.4. So if the congruence $x^2 - 4y^2 - 2 \equiv 0 \pmod 4$ had a solution, then so would $x^2 - 2 \equiv 0 \pmod 4$; and if this congruence had a solution, then it would have a solution in the complete residue set $\{0, 1, 2, 3\}$. But this is not the case by our observations above; therefore,
(1) the congruence $x^2 - 4y^2 - 2 \equiv 0 \pmod 4$ has no integer solutions.

So by the first part of this problem, the equation
(1) $x^2 - 4y^2 - 2 = 0$ (equivalently, the equation $x^2 - 4y^2 = 2$) has no integer solutions.

2 cont. Now we know that, since j and $j+1$ are consecutive integers, one of the two is even. So the product
(2) $j(j+1)$ is also even. The same may be said of $k(k+1)$.

Therefore, the difference $j(j+1) - k(k+1)$ is also even,
(2) and may be written in the form 2ℓ for some integer ℓ.

(1) So we have $m^2 - n^2 = 4(2\ell) = 8\ell$, which is divisible by 8.

3 cont. $m = p_1^{a_1} \cdots p_k^{a_k}$ and $n = p_1^{b_1} \cdots p_k^{b_k}$
(1) for some nonnegative integers $a_1, \ldots, a_k, b_1, \ldots, b_k$.

Now we'll show that, if $c_i = \min(a_i, b_i)$ for $i = 1, \ldots, k$, then the number $\ell = p_1^{c_1} \cdots p_k^{c_k}$ is the GCD of m and n.

(2) First observe that $m = \ell \cdot p_1^{a_1 - c_1} p_2^{a_2 - c_2} \cdots p_k^{a_k - c_k}$
and $n = \ell \cdot p_1^{b_1 - c_1} p_2^{b_2 - c_2} \cdots p_k^{b_k - c_k}$.

(2) So $\ell \mid m$ and $\ell \mid n$; that is, ℓ is a common divisor of m and n. To show that ℓ is the greatest common divisor, first observe that ~~the~~ any common divisor (call it d) of m and n must be
(1) of the form $p_1^{d_1} \cdots p_k^{d_k}$ for some nonnegative integers $d_1, \ldots, d_k$.
~~Now if $d_i > a_i$ for some i, then we have~~
(1) Now we must have $p_i^{d_i} \mid p_1^{a_1} \cdots p_k^{a_k}$ and $p_i^{d_i} \mid p_1^{b_1} \cdots p_k^{b_k}$
(1) for each $i = 1, \ldots, k$, so it follows that $d_i \le a_i$ and $d_i \le b_i$
(1) for each i. So for each i, we have $d_i \le \min(a_i, b_i) = c_i$.
So every common divisor of m and n is of the form
(1) $p_1^{d_1} \cdots p_k^{d_k}$, where $d_i \le c_i$ for each i. Thus $\ell = p_1^{c_1} \cdots p_k^{c_k}$ is the greatest common divisor of m and n.

$$= \frac{(k^2+4k+4)(k+1)^2}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

(3) $$= \left(\frac{(k+1)((k+1)+1)}{2}\right)^2,$$

as desired. Therefore, the case $n=k+1$ holds as well.

(1) Therefore, by the principle of mathematical induction, we have

$$1^3+2^3+\cdots+n^3 = \left(\frac{n(n+1)}{2}\right)^2$$ for every natural number n.