Appendix D

Errata and Addenda

The corrected version of the book, from which the errors listed below have been removed, can be found at

`www.ma.utexas.edu/users/kbi/SDE/C_1.html` .

Erratum at Exercise 2.1.11 on page 52 As stated this exercise runs afoul of the definition of right-continuity on page 23, according to which every single path of a right continuous process is right continuous. It should read

A locally nearly (almost surely) right-continuous process is nearly (respectively almost surely) right-continuous. An adapted process that has locally nearly finite variation nearly has finite variation.

Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (08/04/2005).

Erratum at line +4 on page 54 Replace $Y_2' \stackrel{\text{def}}{=} |X| - |X| \wedge Y_1 = Y_2$ by $Y_2' \stackrel{\text{def}}{=} |X| - |X| \wedge Y_1 \leq Y_2$.

Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (08/04/2005).

Erratum at line +3 on page 58 One cannot have more than one consecutive rationals; so delete the word "consecutive."

Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (08/04/2005).

Erratum at line -9 on page 61 Delete the word "consecutive."

Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (08/04/2005).

Addendum to line +8 on page 62 The superscript (A.8.6) on K_0 says that $K_0^{(A.8.6)}$ is the constant of inequality (A.8.6). We will use this device of pointing to equations etc. throughout. $E_{p,q}^{4.1.2}$ (no parentheses on the superscript) would refer to the constant $E_{p,q}$ appearing in theorem 4.1.2, but this form is never used. However, this was not explained and can lead

the reader astray.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (08/04/2005).

Addendum to Theorem 2.3.6 on page 64 (12/29/2006) The argument lends itself to the following corollary; its proof anticipates the integration theory of dZ, in particular proposition 3.5.2.

Corollary D.1 *Suppose Z is a global L^p-integrator for some $p \in (0,\infty)$. There is an integrable process X of absolute value one so that*

$$\| Z^\star_\infty \|_{L^p} \le C_p^{\star(2.3.6)} \left\| \int X \, dZ \right\|_{L^p} .$$

Proof. With $q = (1+\sqrt{5})/2 > 1$ define inductively, as in the proof of theorem 2.3.6, $T_1 = 0$ and $T_{n+1} = \inf\{s : s > T_n \text{ and } |Z|_s > q|Z|_{T_n}\}$. These stopping times are not elementary, but they do increase to ∞. The estimates of inequality (2.3.7) stay, and at the penultimate line read

$$\|Z^\star_\infty\|_{L^p(\mathbb{P})} \le qL_qK_p \left\| \left\| \int \Big(\sum_n ((T_{n-1}, T_n]] \, \epsilon_n(\tau) \Big) \, dZ \right\|_{L^p(\mathbb{P})} \right\|_{L^p(d\tau)}$$

cont'd as:

$$\le qL_qK_p \sup_\tau \left\{ \left\| \int \Big(\sum_{n=1}^\infty ((T_{n-1}, T_n]] \, \epsilon_n(\tau) \Big) \, dZ \right\|_{L^p(\mathbb{P})} \right\} . \qquad (*)$$

Now the stochastic integral in $(*)$ depends continuously on $\tau \in \{-1,1\}^{\mathbb{N}}$ (see theorem A.8.26); indeed, since $\lceil\!\lceil ((T_N, \infty)) \rceil\!\rceil^*_{Z-p} \xrightarrow[N\to\infty]{} 0$, this integral is the uniform (in τ) limit in $L^p(\mathbb{P})$ of $\int \sum_{n=1}^N \cdots (\tau) \, dZ$, which depends continuously on τ in the discrete space $\{1,-1\}^N$. Hence there is a $\tau \in \{0,1\}^{\mathbb{N}}$ where the supremum in $(*)$ is taken, and $X \overset{\text{def}}{=} \sum_n^\infty ((T_{n-1}, T_n]] \, \epsilon_n(\tau)$ meets the description.

Addendum to line +18 on page 68 It is best to identify the ingredients in the (in)equalities on lines 19 and 20:

Instead of "for $X \in \mathcal{E}_1$ as in equation (2.1.1)"

read "for $X \in \mathcal{E}_1$, f_n, and t_n as in equation (2.1.1)."

Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (08/04/2005).

Erratum at Exercise 2.5.5 on page 72 Read $g^{\mathcal{G}} \overset{\text{def}}{=} \mathbb{E}[g|\mathcal{G}]$ for $g^{\mathcal{G}} \overset{\text{def}}{=} \mathbb{E}[f|\mathcal{G}]$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (12/21/2006).

Erratum at Corollary 2.5.11 on page 74 In the proof, S_A and T_A are generally not elementary, as they take the value $+\infty$ on A^c. So strictly spoken proposition 2.5.10 is not applicable; the displayed equation should be

$$
\begin{aligned}
0 \le \mathbb{E}\Big[\int (\!(S_A \wedge T, T_A \wedge T]\!]\, dM\Big] &= \mathbb{E}\Big[M_{T_A \wedge T} - M_{S_A \wedge T}\Big] \\
= \mathbb{E}\Big[\big(M_T - M_S\big) \cdot 1_A\Big] &= \mathbb{E}\Big[\Big(\mathbb{E}\big[M_T|\mathcal{F}_S\big] - M_S\Big) \cdot 1_A\Big] .
\end{aligned}
$$

Thanks to **Oliver Diaz–Espinoza**, `odiaz@math.mcmaster.ca` (09/18/06)

Erratum at line -9 on page 76 The subscript $U \wedge t$ ended up in the wrong place in this and the next line. The displayed equations should read:

$$
\begin{aligned}
\mathbb{P}\Big[M^S > \lambda\Big] &\le \lambda^{-1} \cdot \int_{[U \le t]} |M_U|\, d\mathbb{P} = \lambda^{-1} \cdot \int_{[U \le t]} |M_{U \wedge t}|\, d\mathbb{P} \\
\text{by corollary 2.5.11:} \quad &\le \lambda^{-1} \cdot \int_{[U \le t]} \mathbb{E}\big[|M_t| \;\big|\; \mathcal{F}_{U \wedge t}\big]\, d\mathbb{P} \\
&= \lambda^{-1} \cdot \int_{[M^S > \lambda]} |M_t|\, d\mathbb{P} \le \lambda^{-1} \cdot \int_{[M_t^\star > \lambda]} |M_t|\, d\mathbb{P} .
\end{aligned}
$$

Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (01/23/2006).

Addendum to Exercise 2.5.32 on page 86 The boundedness and the right-continuity of $S_{\bullet}$ are not needed.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (08/31/2005).

Erratum at Exercise 3.3.3 on page 109 Z should be replaced by Z^t in the displayed equation.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (08/31/2005).

Addendum to Observation 3.4.1 on page 110 (08/31/2005) "Uniformly continuous" means "$\mathcal{E}$-uniformly continuous, of course, in the last sentence.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (08/31/2005).

Erratum at Exercise 3.4.9 on page 113 The last line of part (i) should end with "(take $\mathcal{D} = \mathfrak{C}$)," just as the words before it imply.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (08/31/2005).

Addendum to end of section 3.4 on page 115 If ν is a measure then the dual of $\mathfrak{L}^1[\nu]$ is $\mathfrak{L}^\infty[\nu]$ and the dual of $\mathfrak{L}^p[\nu]$ is $\mathfrak{L}^{p'}[\nu]$, $1 < p < \infty$, $p' = p/(p-1)$. It is natural to ask for a similar characterization of the dual of $\mathfrak{L}^1[\|\ \|^*]$, when $\|\ \|^*$ is a homogeneous mean.

Let us write $\mathfrak{L}$ for $\mathfrak{L}^1[\| \ \|^*]$, $\mathfrak{L}'$ for its dual and $\langle \ | \ \rangle$ for their pairing. If $\mu \in \mathfrak{L}'$ then $F \mapsto \langle F|\mu\rangle$ is clearly a measure on the bounded functions $\mathfrak{L}_b \subset \mathfrak{L}$ satisfying

$$|\langle F|\mu\rangle| \le \| F \|^* \cdot \| \mu \|_{\mathfrak{L}'} \ , \qquad f \in \mathfrak{L} \ ,$$

from which we see that μ has finite variation, is σ-additive, and vanishes on $\| \ \|^*$-negligible functions. In other words, there is an obvious identification of $\mathfrak{L}'$ with the space of σ-additive measures μ of finite variation on $\mathfrak{L}_b$ that vanish on $\| \ \|^*$-negligible functions and have

$$\|\mu\|_{\mathfrak{L}'} = \sup \left\{ \int F \, d\mu : F \in \mathfrak{L} \ , \ \|F\|^* \le 1 \right\} < \infty \ ,$$

and
$$\langle F|\mu\rangle = \int F \, d\mu \ , \qquad F \in \mathfrak{L} \ .$$

Let us now simplify the situation a little by assuming that $\| \ \|^*$ ***is σ-finite***; that is to say, there is a countably collection of integrable sets that cover the ambient space.

Lemma D.2 (Control Measure) *There exists in $\mathfrak{L}'$ a positive measure ν on $\mathfrak{L}_b$ with $\| \nu \|_{\mathfrak{L}'} = 1$ that has exactly the same negligible sets and functions as $\| \ \|^*$, a* ***control measure*** *for $\| \ \|^*$.*

Proof. Consider pairs (A, μ_A) consisting of a $\| \ \|^*$-non-negligible $\| \ \|^*$-integrable set A and a positive σ-additive measure μ_A that satisfies

$$|\mu_A(F)| \le \| F \|^* \quad \text{and} \quad \mu_A(F) = 0 \Longleftrightarrow \| F \|^* = 0 \qquad \forall F \in \mathfrak{L} \ .$$

A maximal collection of such pairs with mutually disjoint first entries is at most countable, so we write it $\{(A^{(1)}, \mu_{A^{(1)}}), \ldots\}$. The complement B of $\bigcup_k A_k$ is $\| \ \|^*$-negligible; if it were not, then the Hahn–Banach theorem A.2.25 would provide a measure $\mu \in \mathfrak{L}'$ with $\mu(B) \ne 0$, which could be chosen positive and having $\| \mu \|_{\mathfrak{L}'} \le 1$. Let $\{B_1, B_2, \ldots\}$ be a maximal collection of disjoint integrable μ-negligible subsets of B with $\| B_i \|^* > 0$. Setting $A^{(0)} \stackrel{\text{def}}{=} B \setminus \bigcup_i B_i$ and $\mu_{A^{(0)}} \stackrel{\text{def}}{=} A^{(0)} \mu$ would produce a pair $(A^{(0)}, \mu_{A^{(0)}})$ that could be adjoined to the supposedly maximal collection $\{(A^{(1)}, \mu_{A^{(1)}}), \ldots\}$. Thus, indeed, $\| B \|^* = 0$. Now set $\nu_0 \stackrel{\text{def}}{=} \sum 2^{-k} \mu_{A^{(k)}}$. Clearly ν_0 and $\| \ \|^*$ have exactly the same negligible sets, and $\| \nu_0 \|_{\mathfrak{L}'} \le 1$. A suitable scalar multiple ν of ν_0 will also have $\| \nu \|_{\mathfrak{L}'} = 1$. ∎

We fix now a control measure $\nu \in \mathfrak{L}'_{1+}$ and return to the characterization of $\mathfrak{L}'$. If $\mu \in \mathfrak{L}'$ then clearly μ is absolutely continuous with respect to ν, so there exists a Radon–Nikodym derivative $F'_\mu \stackrel{\text{def}}{=} d\mu/d\nu$:

$$\langle F|\mu\rangle = \int F \, F'_\mu \, d\nu \ , \qquad F \in \mathfrak{L} \ ,$$

and
$$\left\| F'_\mu \right\|' \stackrel{\text{def}}{=} \sup \left\{ \int F \, F'_\mu \, d\nu : \|F\|^* \le 1 \right\} = \|\mu\|_{\mathfrak{L}'} < \infty \ . \qquad (*)$$

In other words, $\mathfrak{L}'$ can be identified with the space of all $\|\ \|^*$-measurable functions F' with $\|F'\|' < \infty$, where $(*)$ defines the dual norm $\|\ \|'$ and the pairing is $(F, F') \mapsto \langle F|F'\rangle \stackrel{\text{def}}{=} \int F\, F'\, d\nu$.

Theorem D.3 *A uniformly integrable subset of* $\mathfrak{L} \stackrel{\text{def}}{=} \mathfrak{L}^1[\|\ \|^*]$ *is relatively weakly compact.*

Proof. Recall that a subset $\mathfrak{F} \subset \mathfrak{L}$ is uniformly integrable if for every $\epsilon > 0$ there exists an integrable function G_ϵ such that the distance of any $F \in \mathfrak{F}$ from the order interval

$$[-G_\epsilon, G_\epsilon] \stackrel{\text{def}}{=} \{\widehat{F} \in \mathfrak{L} : -G_\epsilon \le \widehat{F} \le G_\epsilon\}$$

is less than ϵ, in other words, if

$$dist(F, [-G_\epsilon, G_\epsilon]) \stackrel{\text{def}}{=} \inf\left\{\|F - \widehat{F}\|^* :\ \widehat{F} \in [-G_\epsilon, G_\epsilon]\right\}$$

is less than ϵ. The previous infimum is actually taken at the function

$$\widehat{F}_\epsilon \stackrel{\text{def}}{=} -G_\epsilon \vee F \wedge G_\epsilon\ .$$

A uniformly integrable set $\mathfrak{F}$ is evidently bounded, with

$$\sup\{\|F\|^* : F \in \mathfrak{F}\} \le M \stackrel{\text{def}}{=} \inf_{\epsilon > 0} \|G_\epsilon\|^* + \epsilon\ .$$

It is easily seen that the convex hull of a uniformly integrable family is uniformly integrable and that so is the closure of the latter, which is $\sigma(\mathfrak{L}, \mathfrak{L}')$-closed. For the proof of the theorem we may therefore assume that we are facing a convex weakly closed uniformly integrable set $\mathfrak{F} \subset \mathfrak{L}$, which must be shown to be $\sigma(\mathfrak{L}, \mathfrak{L}')$-compact.

Let then $\mathfrak{U}$ be an ultrafilter on $\mathfrak{F}$. For every $F' \in \mathfrak{L}'$, $\langle \mathfrak{U}|F'\rangle$ is then an ultrafilter on the compact interval $\big[-M\|F'\|', M\|F'\|'\big]$ and has a limit there, which we shall denote by $\langle \eta|F'\rangle$. Clearly $F' \mapsto \langle \eta|F'\rangle$ is linear.

The fact that
$$\begin{aligned} |\langle F|F'\rangle| &\le |\langle \widehat{F}_\epsilon|F'\rangle| + |\langle F - \widehat{F}_\epsilon|F'\rangle \\ &\le \left|\int \widehat{F}_\epsilon\, F'\, d\nu\right| + \epsilon \\ &\le \big\|G_\epsilon \cdot |F'|\big\|^* + \epsilon \end{aligned}$$

for $F \in \mathfrak{F}$ and $F' \in \mathfrak{L}'_1$ implies that in the limit

$$|\langle \eta|F'\rangle| \le \big\|G_\epsilon \cdot |F'|\big\|^* \cdot \big\|F'\big\|' + \epsilon \cdot \big\|F'\big\|'\ , \qquad F' \in \mathfrak{L}'\ ,$$

from which it is easily seen that $F' \mapsto \langle \eta|F'\rangle$ is a σ-additive measure of finite variation on $\mathfrak{L}'$ and is absolutely continuous with respect to ν. Indeed, let $\mathfrak{L}' \ni F'_n \downarrow 0$ ν-almost surely. Given a $\delta > 0$ we choose first $\epsilon > 0$ so that $\epsilon \cdot \|F'_1\|' < \delta/2$ and then, using the Dominated Convergence

Theorem, N so large that $\| G_\epsilon \cdot |F'_n| \|^* < \delta/\big(2\| F'_1 \|'\big)$ for $n \geq N$, thus showing that $\langle \eta | F'_n \rangle \xrightarrow[n\to\infty]{} 0$. There exists therefore a Radon–Nikodym derivative $H \stackrel{\text{def}}{=} d\eta/d\nu$: $\langle \eta | F' \rangle = \int H\, F'\, d\nu$ for $F' \in \mathfrak{L}'$. The very definition of η reads

$$\lim_{F \in U \in \mathfrak{U}} \langle F | F' \rangle = \langle H | F' \rangle \qquad \forall F' \in \mathfrak{L}'$$

and shows that H is the $\sigma(\mathfrak{L}, \mathfrak{L}')$-limit of $\mathfrak{U}$. ∎

Erratum at line +3 on page 122 The limit is missing.
Please read "… then $f^m \cdot [\![T]\!]$ converges to $f \cdot [\![T]\!]$ Z–0-a.e.…"
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (10/07/2005).

Erratum at Definition 3.5.17 on page 122 In the penultimate line read "any set of strictly positive probability" for "any set of positive probability."

Erratum at Equation (3.6.1) on page 123 This equation should read

$$[\![F]\!]^{**} = \begin{cases} \sup\{[\![X]\!]^* : \ X \in \mathcal{E}_+ \,, \ X \leq F\} & \text{if } F \in \mathcal{E}_+^\uparrow \\ \inf\{[\![H]\!]^{**} : \ |F| \leq H \in \mathcal{E}_+^\uparrow\} & \text{for arbitrary } F. \end{cases} \tag{D.1}$$

Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (10/24/2005).

Erratum at Exercise 3.6.19 on page 129 For "step function over $\mathcal{F}_\infty$" read "step functions over $\mathcal{F}_\infty$."
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (10/24/2005).

Erratum at line -7 on page 136 Replace "precisely where Z jumps" by "only where Z jumps." [X could vanish.]
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (10/24/2005).

Erratum at line -4 on page 149 The formula should read

$$\Big[\!\Big[\sqrt{{}^j[X{*}Z, X{*}Z]_\infty} \Big]\!\Big]_{L^p} \leq K_p^{p\wedge 1} \cdot [\![X]\!]^*_{Z\text{–}p} \,.$$

Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (1/23/2006).

Erratum at line -6 on page 151 Replace the line by
Consequently, $\| \, |\sigma[Y] - \sigma[Z]|^\star_T \|_{L^p} \leq \| S[Y-Z]^\star_T \|_{L^p} \leq K_p^{(3.8.6)} \, \lceil\!\lceil (Y-Z)^T \rceil\!\rceil_{\mathcal{I}^p}$
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (1/23/2006).

Erratum at Exercise 3.8.14 on page 152 The summations over k both in the statement and in the answer should extend over $0 \le k < \infty$ rather than $0 < k < \infty$.
Thanks to **Roger Sewell**, <rfs@cambridgeconsultants.com> (1/23/2006).

Erratum at line +10 on page 153 There is a spurious right parenthesis before the word “against.”
Thanks to **Roger Sewell**, <rfs@cambridgeconsultants.com> (01/23/2006).

Erratum at Exercise 3.8.20 on page 155 In line 3 replace $[Y,Z]^{jT}$ with ${}^j[Y,Z]^T$.
Thanks to **Roger Sewell**, <rfs@cambridgeconsultants.com> (02/23/2006).

Erratum at line +9 on page 157 (01/23/2006) Thanks to **Roger Sewell**, <rfs@cambridgeconsultants.com>, who noticed that this proof is totally garbled and suggested how to fix it. It should read as follows:
Let $T' \le T$ be a bounded stopping time such that V is bounded on $[\![0, T']\!]$ (corollary 3.5.16). Since by exercise 3.8.12 ${}^c[M,V] = 0$, taking the difference of the representations of $M \cdot V$ at the times $T' \vee S$ and S that are given by proposition 3.8.22 results in

$$M_{T'\vee S} V_{T'\vee S} - M_S V_S = \int_{S+}^{T'\vee S} M_{.-}\, dV + \int_{S+}^{T'\vee S} V\, dM \; .$$

The term on the far right has expectation 0. Take the expectation in the displayed formula and let $T' \uparrow T$: the DCT gives the claim.

Erratum at Exercise 3.8.24 on page 157 (02/23/2006) See the answer.

Erratum at Theorem 3.9.1 on page 158 In the details to the proof on page 28 of the Answers there are two typos: in line 3 of the answer, $\overline{\Phi_{;\eta}}_t \overline{\Psi_{;\eta}}_t d[Z^\eta, Z^\theta]^c_t$ should be replaced by $\overline{\Phi_{;\eta}}_t \overline{\Psi_{;\theta}}_t d[Z^\eta, Z^\theta]^c_t$; and three lines later $\overline{\Psi'_\theta}$ by $\overline{\Psi_{;\theta}}$.
Thanks to **Roger Sewell**, <rfs@cambridgeconsultants.com> (02/23/2006).

Erratum at line -1 on page 160 The M in the exponent should be an N.
Thanks to **Roger Sewell**, <rfs@cambridgeconsultants.com> (01/23/2006).

Erratum at Equation (3.9.6) on page 162 Equation (3.9.5) on page 162 should read

$$M' = \Big(M'_0 - G_{\bullet-}{*}[M', G']\Big) + \Big(G_{\bullet-}{*}(M'G') - (M'G)_{\bullet-}{*}G'\Big) \,, \qquad \text{(D.2)}$$

and equation (3.9.6) should read

$$M + G'_{\bullet-}{*}[M, G] = M_0 + G'_{\bullet-}{*}(MG) - (MG')_{\bullet-}{*}G \,, \qquad \text{(D.3)}$$

every one of the processes on the right in (D.3) being a local $\mathbb{P}'$-martingale.

Erratum at Equation (3.10.3) on page 175 The equation between (3.10.2) and (3.10.3) should read

$$\lceil\!\lceil \check{F}{*}\zeta \rceil\!\rceil_{\mathcal{I}^p} = \llbracket \check{F} \rrbracket^*_{\zeta\text{–}p} = \llbracket \check{F} \rrbracket_{\mathfrak{L}^1[\zeta\text{–}p]}$$

Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (09/09/2006).

Erratum at line -5 on page 181 For $n + \|H\|_\infty$ read $(n{+}1)\|H/h_0\|_\infty$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (02/16/2007).

Erratum at Proposition 3.10.10 on page 182 Φ must be thrice continuously differentiable for this to make sense. Also, there is the subscript ";" missing in the last line.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (09/09/2006).

Erratum at Equation (4.1.2) on page 187 This and the previous inequality are proved only for $\alpha \in (0, 1 \wedge 4\alpha_1)$, not for all $\alpha \in (0,1)$. The same problem arises in the proof on page 207 of proposition 4.1.12 (iii).
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (10/09/2006).
A slight change in the definition of $g' = d\mathbb{P}'/d\mathbb{P}$ overcomes this problem; it is incorporated in the corrected version of the book at

`www.ma.utexas.edu/users/kbi/SDE/C_1.html` .

(I managed to include more mistakes in the first correction and am profoundly grateful to Dr. Sewell for finding them and pointing them out to me.)

Erratum at line +20 on page 188 Replace "comost" by "most."
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (09/09/2006).

Erratum at line +1 on page 188 This line should read as follows:
The complement $G \stackrel{\text{def}}{=} [T = \infty] = [Z^\star_\infty \le \lceil\!\lceil Z \rceil\!\rceil_{[\alpha/2]}]$ has $\mathbb{P}[G] \ge 1 - \alpha/2$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (09/09/2006).

Addendum to Theorem 4.1.2 on page 191 (Juli 2003) If the estimates (4.1.6) and (4.1.9) on page 191 are not needed one can have the following result:

Theorem D.4 *Suppose* $\{Z^{(i)} : i \in \mathbb{N}\}$ *is a countable collection of* $L^0(\mathbb{P})$*-integrators. There exists a probability* $\mathbb{P}'$ *equivalent with* $\mathbb{P}$ *on* $\mathcal{F}_\infty$ *and having a bounded Radon–Nikodym derivative* $d\mathbb{P}'/d\mathbb{P} > 0$ *so that* $Z^{(i)}$ *is an* $L^p(\mathbb{P}')$*-integrator for all* $p < \infty$ *and all* $i \in \mathbb{N}$*. In fact, given any sequence* (T_n) *of almost surely finite stopping times that increases almost surely without bound,* $\mathbb{P}'$ *can be chosen so that every one of the stopped processes* $Z^{(i)T_n}$ *is a global* $L^p(\mathbb{P}')$*-integrator for every* $p < \infty$ *and every* $i \in \mathbb{N}$*.*

For the proof[1] an auxiliary result is needed.

Lemma D.5 (Mokobodzki–Dellacherie) *(i) Let* K *be a convex subset of* $L^0(\mathcal{F}, \mathbb{P})$ *that satisfies*
(a) $0 \in K$[2] *and* $K_- \stackrel{\text{def}}{=} \{k \wedge 0 : k \in K\} \subset L^1(\mathcal{F}, \mathbb{P})$ *and*
(b) $K_+ \stackrel{\text{def}}{=} \{k \vee 0 : k \in K\}$ *is bounded in* $L^0(\mathbb{P})$*.*
Then there exists a probability $\mathbb{P}'$ *that is equivalent with* $\mathbb{P}$ *on* $\mathcal{F}$ *and has bounded Radon–Nikodym derivative* $d\mathbb{P}'/d\mathbb{P} > 0$ *such that*

$$\sup\left\{ \int k \, d\mathbb{P}' \ : \ k \in K \right\} < \infty \,. \tag{D.4}$$

(ii) Suppose now $\mathcal{K}$ *is a countable collection of convex subsets of* $L^0(\mathbb{P})$ *each of which satisfies (a) and (b) above. Again there exists a probability* $\mathbb{P}'$ *equivalent with* $\mathbb{P}$ *and having bounded Radon–Nikodym derivative* $d\mathbb{P}'/d\mathbb{P} > 0$*, so that inequality (D.4) is satisfied on every single* $K \in \mathcal{K}$*.*

Proof. Let $\qquad K_0 \stackrel{\text{def}}{=} \{f \in L^1(\mathbb{P}) : \exists k \in K \text{ with } f \le k\}$

and $\qquad \overline{K_0} \stackrel{\text{def}}{=}$ the closure of K_0 in $L^1(\mathbb{P})$.

$\overline{K_0}$ is again convex, and every function $k \in K$ is the pointwise supremum of the functions $k \wedge n \in K_0$. Assumption (b) has the consequence that

$$\forall\, A \in \mathcal{F} \text{ with } \mathbb{P}[A] > 0 \quad \exists\, c \in \mathbb{R}_+ \text{ such that } cA \notin \overline{K_0} \,. \tag{D.5}$$

Indeed, as K_+ is bounded in $L^0(\mathbb{P})$ there is a $c \in \mathbb{R}_+$ with $\sup_{k \in K_+} \mathbb{P}[k \ge c] \le \mathbb{P}[A]/2$. Then clearly $\sup_{k \in \overline{K_0}} \mathbb{P}[k \ge c] \le \mathbb{P}[A]/2$ as well, which implies that cA cannot belong to $\overline{K_0}$.

Yan[1] has shown that condition (D.5) is necessary and sufficient for the conclusion of (i). We show the sufficiency. To this end let, for any $G \in L^\infty_+$,

$$c[G] \stackrel{\text{def}}{=} \sup\{\mathbb{E}[G \cdot k] : k \in K\} = \sup\{\mathbb{E}[G \cdot k] : k \in \overline{K_0}\} \,,$$

and let $\mathcal{G} \stackrel{\text{def}}{=} \{G \in L^\infty_+ : c(G) < \infty\}$ and $z \stackrel{\text{def}}{=} \inf\{\mathbb{P}[G = 0] : G \in \mathcal{G}\}$.

[1] We rely heavily on the results and arguments of Jia–an Yan's article in Sem. Prob. XIV, page 220, where further literature is cited.
[2] If $K \subset L^1(\mathbb{P})$ then this condition is superfluous, as it can be had by a simple translation.

There exists a sequence $G_n \in \mathcal{G}$ with $z = \lim_n \mathbb{P}[G_n]$. With $\lambda_n > 0$ chosen so that $\sum \lambda_n \cdot (c[G_n] + \|G_n\|_\infty) < \infty$, clearly $G' \stackrel{\text{def}}{=} \sum \lambda_n G_n \in \mathcal{G}$ and $z = \mathbb{P}[G' = 0]$. A suitable multiple of $G' \cdot \mathbb{P}$ will be the desired probability $\mathbb{P}'$ satisfying (D.4), provided that $z = 0$. We argue by contradiction. If $z > 0$ then there is a constant $c > 0$ with $c[G' = 0] \notin \overline{K_0}$. The Hahn–Banach theorem provides a continuous linear functional g' in the dual $L^\infty(\mathbb{P})$ of $L^1(\mathbb{P})$ so that

$$\sup_{k \in \overline{K_0}} \int k \cdot g' \, d\mathbb{P} < \int c[G = 0] \cdot g' \, d\mathbb{P} \, . \tag{D.6}$$

Since $0 \in K$, $\overline{K_0}$ contains every negative bounded measurable function, in particular $n \cdot (g' \wedge 0)$, $n \in \mathbb{N}$, which when substituted for k in (D.6) shows that $\sup_n n \cdot \int (g'_-)^2 \, d\mathbb{P} < \infty$ so that g' must be almost surely positive. Therefore $g' \in \mathcal{G}$. But then $G' + g'$ belongs to $\mathcal{G}$, and since the integral on the right in inequality (D.6) is strictly positive, $\mathbb{P}[G' + g' = 0] < \mathbb{P}[G' = 0]$, which contradicts the definition of z. This proves (i).

An aside. Suppose we know in advance that $K \subset L^\infty(\mathbb{P})$. Define K_0 instead as $K - L^\infty_+$ and denote by $\overline{K_0}^p$ the closure of K_0 in $L^p(\mathbb{P})$, and define in the previous argument $\overline{K_0}$ instead as the intersection of the $\overline{K_0}^p$, $p < \infty$. Then Theorem D.2 persists. A slight change to the argument occurs at inequality (D.6). From $c[G' = 0] \notin \overline{K_0}$ we can now only conclude that $c[G' = 0] \notin \overline{K_0}^p$ for some $p < \infty$, so that the function g' appearing in inequality (D.6) can only be claimed to be in $L^{p'}(\mathbb{P})$. But then $g' \wedge n$ will satisfy the same inequality for sufficiently large $n \in \mathbb{N}$, and, replacing g' by it, we can continue the argument as above.

Proof of (ii) (P.–A. Meyer[1]): Let us count $\mathcal{K} = \{K_n : n \in \mathbb{N}\}$. Let $c_{n,m} > 0$ be such that $\mathbb{P}[k > c_{n,m}] \le 2^{-n}/m$ for all $k \in K_n$ and all $m \in \mathbb{N}$; then choose $\lambda_n > 0$ so that $c_m \stackrel{\text{def}}{=} \sum_n \lambda_n c_{n,m} < \infty \quad \forall\, m$. The sets $L_N \stackrel{\text{def}}{=} \sum_{n \le N} \lambda_n K_n$ are again convex and satisfy (a) and (b), and so does their union L. To see that L satisfies (b), observe that every $\ell \in L$ is a finite sum $\ell = \sum_n \lambda_n k_n$ with $k_n \in K_n$, so

$$\mathbb{P}[\ell \ge c_m] \le \sum_n \mathbb{P}[k_n \ge c_{m,n}] \le \sum_n 2^{-n}/m = 1/m \quad \forall\, m \ :$$

the positive part L_+ of L is indeed bounded in $L^0(\mathbb{P})$, and the probability $\mathbb{P}'$ provided by part (i) for L meets the description of (ii). ∎

Proof of Theorem D.4. Lemma D.5 (ii), applied to the sets

$$K_n \stackrel{\text{def}}{=} \left\{ \sum_{i=1}^n \int_0^{T_n} X^{(i)} \, dZ^{(i)} : X^{(i)} \in \mathcal{E}, |X^{(i)}| \le 1 \right\} ,$$

provides a probability equivalent with $\mathbb{P}$ and having bounded Radon–Nikodym derivative with respect to which $(Z^{(1)}, \ldots, Z^{(n)})$ is an L^1-integrator. Actually, using Theorem 4.1.2 on page 191, we can say more: At every time T_n there is a probability $\mathbb{P}^n \approx \mathbb{P}$ so that the stopped processes $Z^{(1)T_n}, \ldots, Z^{(n)T_n}$ are global $L^n(\mathbb{P}^n)$-integrators. This implies that the set $\{\sum_{i=1}^{n}\int_0^{T_n} X^{(i)}\,dZ^{(i)} : X^{(i)}\in\mathcal{E}, |X^{(i)}|\le 1\}$ is bounded in $L^n(\mathbb{P}^n)$, or again, that the set K_n of convex combinations of random variables the form $|\sum_{i=1}^{n}\int_0^{T_n} X^{(i)}\,dZ^{(i)}|^n$, $X\in\mathcal{E}, |X|\le 1$, is bounded in $L^1_+(\mathbb{P}^n)$. K_n is then *a fortiori* bounded in $L^0(\mathbb{P}^n) = L^0(\mathbb{P})$ and its negative part $(K_n)_- = \{0\}$ belongs to $L^1(\mathbb{P})$. The probability $\mathbb{P}'\approx\mathbb{P}$ produced by Lemma D.5 (ii) clearly meets the description. ∎

A similar argument shows that an L^0-random measure is an L^2-random measure for a suitable equivalent probability. Note again that these arguments destroy any estimate of $\mathbb{P}$ in terms of $\mathbb{P}'$ as in inequalities (4.1.6) and (4.1.9) on page 191.

Erratum at Exercise 4.1.3 on page 192 The last line of part (i) should read

$$\|f\|_{L^r(\mu)} \le c^{p/(rq)}\cdot\|f\|_{L^{rq/p}(d\mu/g)}\ .$$

Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (09/09/2006).

Erratum at Exercise 4.1.6 on page 193 We cannot expect subadditivity of $\mathcal{I}\mapsto\eta_{p,q}(\mathcal{I})$ when $p<1$, simply because the mean $f\mapsto\|f\|_{L^p(\mu)}$ appearing in inequality (4.1.12) is not subadditive then (see exercise A.8.2 on page 448). For $0<p<q<\infty$ the correct inequality is

$$\eta_{p,q}(\mathcal{I}+\mathcal{I}')\le 2^{0\vee(1-q)/q}\cdot 2^{0\vee(1-p)/p}\times\big[\eta_{p,q}(\mathcal{I})+\eta_{p,q}(\mathcal{I}')\big]\ .$$

Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (09/09/2006).

Erratum at line +18 on page 194 Replace $/4$ by $/q^2$ in the last exponent:

$$k_{x_1,\ldots,x_n} = \Big(\int\Big(\sum_{1\le\nu\le n}|\mathcal{I}x_\nu|^q\Big)^{p/q}d\mu\Big)^{(p-q)/q}\cdot\Big(\sum_{1\le\nu\le n}|\mathcal{I}x_\nu|^q\Big)^{p(q-p)/q^2}$$

Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (09/09/2006).

Erratum at line +4 on page 198 Replace this line by

$$a=\sup\{\langle g|f^*\rangle : g\in B_a(0)\}\le\langle f|f^*\rangle\le a\ ,$$

Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (09/09/2006).

Erratum at line +3 on page 199 Replace $\|f\|_{L^2(\ell^\infty)}$ by $\|f\|_{L^2(\tau,\ell^\infty)}$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (09/14/2206).

Erratum at line +6 on page 200 In that line $[Q_1 + Q_s]$ should be replaced by $[Q_1 + Q_2]$. Line 7 and the following sequence of (in)equalities should be replaced by the following:

The first term Q_1 can be bounded using Jensen's inequality (A.3.10) for the probability $|\phi_\delta|/\gamma \cdot \tau$, where $\gamma \stackrel{\text{def}}{=} \int |\phi_\delta(s)|\,\tau(ds) \le 1/\sqrt{\delta}$:

$$\begin{aligned}
\int \|(f\star\phi_\delta)(t)\|^2_{\ell^\infty}\,\tau(dt) &= \int \Big\|\int f(st)\phi_\delta(s)\,\tau(ds)\Big\|^2_{\ell^\infty}\,\tau(dt) \\
\text{by A.3.28:} \qquad &\le \int \Big(\int \|f(st)\|_{\ell^\infty}|\phi_\delta(s)|\,\tau(ds)\Big)^2\,\tau(dt) \\
&= \gamma^2 \int \Big(\int \|f(st)\|_{\ell^\infty}|\phi_\delta(s)|/\gamma\,\tau(ds)\Big)^2\,\tau(dt) \\
&\le \gamma^2 \int\int \|f(st)\|^2_{\ell^\infty}|\phi_\delta(s)|/\gamma\,\tau(ds)\,\tau(dt) \\
&= \gamma^2 \int \|f(t)\|^2_{\ell^\infty}\,\tau(dt) \int |\phi_\delta(s)|/\gamma\,\tau(ds) \\
&= \gamma^2 \int \|f(t)\|^2_{\ell^\infty}\,\tau(dt) \le \delta^{-1} \int \|f(t)\|^2_{\ell^\infty}\,\tau(dt)\,,
\end{aligned}$$

so that $$\|f\star\phi_\delta\|_{L^2(\tau,\ell^\infty)} \le \frac{1}{\sqrt{\delta}} \cdot \|f\|_{L^2(\tau,\ell^\infty)}\,. \tag{4.1.24}$$

Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (09/14/2206).

Erratum at line +2 on page 201 The displayed equation should read

$$\Big\|\,\|\mathcal{I}W f\star\phi_\delta\|_{L^2(\tau)}\Big\|_{L^p(\mu)} \le \delta \cdot \eta_{p,2}(\mathcal{I}) \cdot \|f\|_{L^2(\tau,\ell^\infty)}$$

Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (09/14/2206).

Erratum at lines 9–18 on page 203 Lines 9 and 11 have the Kronecker delta missing. The displayed (in)equalities should read

$$\begin{aligned}
\text{and} \quad \Phi''_{\mu\nu}(m) &= 2(q-1)Q^{\frac{2-q}{q}}(m) \cdot \big|m^\mu\big|^{q-2} \cdot \delta_{\mu\nu} \\
&\quad + 2(2-q)Q^{\frac{2-2q}{q}}(m) \cdot \big|m^\mu\big|^{q-1}\operatorname{sgn}(m^\mu)\big|m^\nu\big|^{q-1}\operatorname{sgn}(m^\nu) \qquad (*) \\
&\le 2(q-1)Q^{\frac{2-q}{q}}(m) \cdot \big|m^\mu\big|^{q-2} \cdot \delta_{\mu\nu}\,,
\end{aligned}$$

Also, we should have recalled the conventional order on symmetric matrices: $A \le B$ if $B - A$ is positive-semidefinite, and we should have written an explicit summation over μ in the same upper position in lines 16–18.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (10/09/2006).

Erratum at Equation (4.1.31) on page 203 The last line has the term $\sum_{\mu} \left|M_\lambda^\mu\right|^{q-2}$ missing. It should read

$$\times\mathbb{E}'\left[\int_0^\infty Q^{\frac{2-q}{q}}(M_\lambda)\sum_\mu \left|M_\lambda^\mu\right|^{q-2} X_\eta^\mu X_\theta^\mu d[\widetilde{Z}^\eta, \widetilde{Z}^\theta]\right] d\lambda\ .$$

Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (10/09/2006).

Erratum at Equation (4.1.34) on page 205 Using the corrected version of exercise 4.1.6 on page 193 (see erratum above), inequality (4.1.34) turns into

$$\eta_{p,q}\Big(\int \cdot\ d\boldsymbol{Z}\Big) \le 2^{1\vee 1/p}(\sqrt{q-1}+1)D_{p,2}\ \lceil\!\lceil \boldsymbol{Z} \rceil\!\rceil_{\mathcal{I}^p[\mathbb{P}]}$$

or $$D_{p,q,d} \le 2^{1\vee 1/p}(1+\sqrt{q-1})D_{p,2} \le 3\cdot 2^{1+4/p}\cdot(1+\sqrt{q-1}) \qquad (4.1.34)$$

and inequality (4.1.35) reads, for suitable $d_{p,q}$,

$$D_{p,q,d} \le \sqrt{d}\cdot d_{p,q}\cdot D_{p,2}\ . \qquad (4.1.35)$$

Erratum at Equation (4.1.35) on page 205 Replace g by g', so that this line now reads "$\left\| g' \right\|_{L^{2/(q-2)}(\mathbb{P}')} < 2^{q/2}$ lead to"
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (10/09/2006).

Erratum at line -3 on page 205 The reference should not be to exercise A.8.31 but to pages 458–463. The same correction is needed on page 206, line 15–16.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (10/09/2006).

Erratum at Exercise 4.1.13 on page 207 Replace $\sqrt{\alpha}$ by α^2.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (10/09/2006).

Erratum at lines 10 and 12 on page 207 There are exponents q missing. The lines should read

$$k_{x_1,\ldots,x_n} \stackrel{\text{def}}{=} \left[\left|\phi_{x_1,\ldots,x_n}\right|^{1/q} \le C_{[\alpha],q}[\|\mathcal{I}\|_{[\boldsymbol{.}]}]\cdot\Big(\sum_{\nu=1}^n \|x_\nu\|_E^q\Big)^{1/q}\right]$$

and

$$\mathbb{E}[\phi_{x_1,\ldots,x_n}\cdot k_{x_1,\ldots,x_n}] \le C_{[\alpha],q}^q[\|\mathcal{I}\|_{[\boldsymbol{.}]}]\cdot\sum_{\nu=1}^n \|x_\nu\|_E^q\ .$$

Thanks to **Roger Sewell**, <rfs@cambridgeconsultants.com> (10/09/2006).

Erratum at line +1 on page 211 U cannot and need not be bounded.
Thanks to **Roger Sewell**, <rfs@cambridgeconsultants.com> (11/07/2006).

Erratum at line -3 on page 211 Replace N^T by N here and in the first two displayed lines on the next page.
Thanks to **Roger Sewell**, <rfs@cambridgeconsultants.com> (11/07/2006).

Erratum at line -8 on page 212 Read "...same token $(p/2)\cdot S_0^{p-2}\cdot S_0^2 \leq S_0^p$."
Thanks to **Roger Sewell**, <rfs@cambridgeconsultants.com> (11/07/2006).

Erratum at line -3 on page 214 The double–or–die martingale N has $N_\infty = 0$ and $N^*_\infty > 0$ almost surely: the line in question should read

$$\| M^\star_\infty \|_{L^p} \leq p' \cdot \sup_{t<\infty, T\in\mathcal{T}} \left\| M^T_t \right\|_{L^p} \leq p' C_p^{(4.2.3)} \cdot \| S_\infty[M] \|_{L^p} ,$$

where $\mathcal{T}$ is the collection of stopping times reducing M to a martingale.
Thanks to **Roger Sewell**, <rfs@cambridgeconsultants.com> (11/07/2006).

Erratum at line 3 on page 220 For better estimates replace $C_p^{(4.2.4)}$ by $C_p^{(4.2.12)}$, $19p'$ by $17p'$, and $\sqrt{e}p^{3/2}p' \vee 19p$ by $\sqrt{e}p^{3/2}p'$.
Thanks to **Roger Sewell**, <rfs@cambridgeconsultants.com> (11/07/2006).

Erratum at Exercise 4.2.19 on page 220 Read ... X is M–r-integrable.
Thanks to **Roger Sewell**, <rfs@cambridgeconsultants.com> (11/07/2006).

Erratum at Exercise 4.2.22 on page 220 Replace "*with* every $\boldsymbol{X} = (X_i) \in \mathfrak{L}^1[\boldsymbol{M}\text{–}p]$" by "*for* every $\boldsymbol{X} = (X_i) \in \mathfrak{L}^1[\boldsymbol{M}\text{–}p]$."
Thanks to **Roger Sewell**, <rfs@cambridgeconsultants.com> (11/07/2006).

Erratum at Exercise 4.2.23 on page 220 To clarify: "M' is perpendicular to every martingale in $\mathcal{A}^{\|}$" here means that $\mathbb{E}[M'_\infty \cdot M_\infty] = 0 \quad \forall\, M \in \mathcal{A}^{\|}$.
Thanks to **Roger Sewell**, <rfs@cambridgeconsultants.com> (11/07/2006).

Erratum at Exercise 4.2.24 on page 220 Replace $\mathbb{P}' \stackrel{\text{def}}{=} G'\mathbb{P}$ by $\mathbb{P}' \stackrel{\text{def}}{=} G'_\infty\mathbb{P}$ and "eqivalent" by "equivalent."
Thanks to **Roger Sewell**, <rfs@cambridgeconsultants.com> (11/07/2006).

Erratum at Equation (4.3.3) on page 221 Replace 2 by 1 in line two, $C_{p'}^{(4.2.5)} \le 6/\sqrt{p'}$ by $pC_{p'}^{(4.2.5)} \le 6p/\sqrt{p'}$ in line three, 4 by 4.1 in line four, 2 by 1 in line five, 5 by 5.1 in line eight, and $6p$ by $6.5p$ in the last line.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (12/21/2006).

Erratum at line +3 on page 221 Strike line 3 to the end of line 4.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (12/01/2207).

Erratum at line +19 on page 223 Replace the first superscript g by g_n:

$$\lim_{n\to\infty} \mu^t(g_n) = \lim_{n\to\infty} \mu\Big(M_0^{g_n} \cdot [\![0]\!] + M_{.-}^{g_n} \cdot (\!(0,t]\!]\Big) = 0 \ .$$

Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (12/21/2006).

Erratum at line +8 on page 224 There is no assumption on I. So, for "the assumption on I" read "the equality above."
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (12/21/2006).

Erratum at Exercise 4.3.5 on page 225 A ***potential*** really is universally defined as a *positive* supermartingale Z with $\mathbb{E}[Z_t] \xrightarrow[t\to\infty]{} 0$. But even after correcting the omission of the word "positive," the statement given is still false. There are several correct versions of it. To state them succinctly, a definition: A process Z is said to be ***of class (D)*** if the random variables $\{Z_T : T \in \mathfrak{T}[\mathcal{F}_.] \,, T < \infty\}$ are uniformly integrable.

Here is what can be said about our positive supermartingale Z that is right continuous in probability:

1) Z has a càdlàg version – which we substitute for Z forthwith.

2a) For Z to be a local L^1-integrator, with a Doob–Meyer decomposition $Z = \widehat{Z} + \widetilde{Z}$, it is necessary and sufficient that Z be locally of class (D), i.e., that there exist arbitrarily large stopping times U such that the stopped process Z^U is of class (D); in this case $\widehat{Z}$ is evidently decreasing.

2b) For Z to have a Doob–Meyer decomposition $Z = \widehat{Z} + \widetilde{Z}$ with $\widehat{Z}$ of integrable finite variation ($\vert\widehat{Z}\vert_t \in L^1 \quad \forall t < \infty$) and $\widetilde{Z}$ a martingale (rather than merely a local one) it is necessary and sufficient that the stopped process Z^u be of class (D) at all instants $u < \infty$ — this condition is known in the literature as condition (DL).

2c) For Z to have a Doob–Meyer decomposition $Z = \widehat{Z} + \widetilde{Z}$ such that $\widehat{Z}$ is a global L^1-integrator and the martingale part $\widetilde{Z}$ is uniformly integrable it is necessary and sufficient that Z be of class (D).

In this case $Z_\infty = \lim_{t\to\infty} Z_t$ exists almost surely and in L^1-mean; $\widehat{Z}_t$

decreases almost surely and in L^1-mean to $\widehat{Z}_\infty \in L^1$; and $\widetilde{Z}_t \xrightarrow[t\to\infty]{} \widetilde{Z}_\infty \in L^1$. There are proofs of these claims in the answers.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (12/21/2006).

Erratum at line -15 on page 225 Switch $\widehat{Z}$ and $\widehat{Z}'$ so as to get:
Then $M \stackrel{\text{def}}{=} \widetilde{Z} - \widetilde{Z}' = \widehat{Z}' - \widehat{Z}$ is a predictable local martingale ...
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (12/21/2006).

Erratum at Exercise 4.3.6 on page 226 For clarity's sake replace "In the general case" by "For any local L^1-integrator Z."
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (12/21/2006).

Erratum at line -1 on page 226 Replace $\| \int Y \, d¦V¦ \|_{L^p}$ by $\| \int |Y| \, d¦V¦ \|_{L^p}$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (12/21/2006).

Erratum at line -6 on page 227 Replace the factor 4 by 4.1.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (12/21/2006).

Erratum at line +2 on page 228 Maybe "...follow by addition" is a better way of saying this: $\widehat{Z} = Z - \widetilde{Z} \Rightarrow ¦\widehat{Z}¦_{\mathcal{I}^p} \le ¦Z¦_{\mathcal{I}^p} + ¦\widetilde{Z}¦_{\mathcal{I}^p} \le (1 + \widetilde{C}_p)¦Z¦_{\mathcal{I}^p}$, except for $p = 1$, where $\widehat{C}_1 = 1$ is better and comes directly from the definition of $\widehat{Z}$. [Corrected constants appear in an erratum above.]
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (12/21/2006).

Erratum at Exercise 4.3.20 on page 229 Replace $\lceil M^t \rceil_{\mathcal{I}^p}$ by $¦M^t¦_{\mathcal{I}^p}$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (12/21/2006).

Erratum at lines +5, +8, +9, +11 on page 230 Replace T by t.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (12/21/2006).

Erratum at Exercise 4.3.23 on page 231 In the first line of the displayed equation $0 < p < 1$ should be replaced by $1 \le p \le 2$ and the constant 4 by 4.1; and then it is no improvement.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (12/21/2006).

Addendum to the subsection on the previsible bracket on page 231 (Transformation of the Doob–Meyer decomposition under a Change of Mea-

sure) Suppose the local $L^1(\mathbb{P})$-integrator Z has Doob–Meyer decomposition $Z = \widehat{Z} + \widetilde{Z}$, and let $\mathbb{P}'$ be a probability equivalent to $\mathbb{P}$ on $\mathcal{F}_\infty$, as in Lemma 3.9.11 on page 162. There are a uniformly $\mathbb{P}$-integrable $(\mathcal{F}_{\bullet}, \mathbb{P})$-martingale G' and a uniformly $\mathbb{P}'$-integrable $(\mathcal{F}_{\bullet}, \mathbb{P}')$-martingale G such that $\mathbb{P}' = G'_\infty \cdot \mathbb{P}$, $\mathbb{P} = G_\infty \cdot \mathbb{P}$, and $G \cdot G' = 1$.

To make the computation below meaningful, let us assume that Z is a local $L^p(\mathbb{P})$-integrator and that G' is locally bounded in $L^{p'}(\mathbb{P})$, for some $p \in [1, \infty)$. This has the effect that Z is a local $L^1(\mathbb{P}')$-integrator (use corollary 4.4.3 on page 234) and thus has a Doob–Meyer decomposition $Z = \widehat{Z}' + \widetilde{Z}'$ with respect to $\mathbb{P}'$. Here is a computation of $\widehat{Z}'$ and $\widetilde{Z}'$. Applying Equation (D.2) on page 8 with $M = \widetilde{Z}$ gives

$$\begin{aligned}
\widetilde{Z}' &= -G_{\bullet-} * \langle \widetilde{Z}', G' \rangle + \underbrace{\Big(- G_{\bullet-} * \widetilde{[\widetilde{Z}', G']} + G_{\bullet-} * (\widetilde{Z}' G') - \widetilde{Z}' G'_{\bullet-} * G' \Big)} \\
&= -G_{\bullet-} * \langle \widetilde{Z}', G' \rangle + \qquad\qquad \overline{Z} \quad \text{(a local } \mathbb{P}\text{-martingale)} .
\end{aligned}$$

Since in view of exercise 4.3.18 $\langle \widetilde{Z}', G' \rangle = \langle Z, G' \rangle$, we get

$$Z = \widehat{Z} + \widetilde{Z} = \big(\widehat{Z}' - G_{\bullet-} * \langle Z, G' \rangle \big) + \overline{Z} ,$$

whence by the uniqueness of the Doob–Meyer decomposition

$$\widehat{Z} = \big(\widehat{Z}' - G_{\bullet-} * \langle Z, G' \rangle \big) \quad \text{and} \quad \widetilde{Z} = \overline{Z} .$$

This gives the relations

$$\widehat{Z}' = \widehat{Z} + G_{\bullet-} * \langle Z, G' \rangle = \widehat{Z} + G_{\bullet-} * \langle \widetilde{Z}, G' \rangle$$

and $\widetilde{Z}' = \widetilde{Z} - G_{\bullet-} * \langle Z, G' \rangle = \widetilde{Z} - G_{\bullet-} * \langle \widetilde{Z}, G' \rangle$

Erratum at Exercise 4.3.27 on page 232 This exercise is phrased too cavalierly: Φ must be somewhat smooth and $\boldsymbol{Z}_0$, $\boldsymbol{Z}_{\bullet-}$ appear in the expression. The exercise should read

If $\boldsymbol{Z}$ is a vector of L^1-integrators, then $\big(\widehat{\boldsymbol{Z}}, {}^c[Z^\eta, Z^\theta], \widehat{\jmath_{\boldsymbol{Z}}}\big)$ is called the ***characteristic triple*** of $\boldsymbol{Z}$. The expectation of any random variable of the form $\Phi(\boldsymbol{Z}_t)$, $\Phi \in C_b^2$, can be expressed in terms of $\boldsymbol{Z}_0$, $\boldsymbol{Z}_{\bullet-}$ and the characteristic triple.

Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (05/23/2007).

Addendum to Section 4.4 on page 232 (May 2003) It may be well to compare the integration theory described in chapter 3 of this book with the stochastic integral developed in Paul–André Meyer's lectures [74]–[75] and, slightly altered, in Protter's book [92]. Since the latter these days seems to be the preferred reference for stochastic integration with respect to semimartingales that jump, we refer to its notations and presentation in the following comparison.

Suppose $Z = V + M$ is a semimartingale. It gives rise to the indefinite

stochastic integral $\quad X \mapsto X \cdot Z \stackrel{\text{def}}{=} \sum f_i \cdot (Z^{T_{i+1}} - Z^{T_i})$

on the processes $\quad X = \sum f_i \cdot (\!(T_i, T_{i+1}]\!]$, $\qquad (*)$

with stopping times $T_i \le T_{i+1}$, $f_i \in L^\infty(\mathcal{F}_{T_i})$, the sum finite.

It is easy to see that $X \mapsto X \cdot Z$ is a linear continuous map from the vector space $\mathfrak{E}$ of processes as in $(*)$ equipped with the topology *ucp* of uniform convergence on compact intervals in probability, to the vector space $\mathfrak{D}$ of adapted càdlàg processes equipped with the same topology. There exists therefore an extension by continuity to the space $b\mathfrak{L}$ of bounded adapted càglàd processes, which evidently belongs to the *ucp*-closure of $\mathfrak{E}$. This first extension is again denoted by $X \cdot Z$ in [92, pp. 34–45], and is called the ***stochastic integral***, with the qualifier "indefinite" left off (we would call the same thing the indefinite integral and denote it by $X{*}Z$).

Next, if Z happens to be a global L^2-integrator, then among its decompositions into a finite variation part and a local martingale there is a canonical one where the finite variation part is previsible, to wit, the Doob–Meyer decomposition $Z = \widehat{Z} + \widetilde{Z}$. In this case there is the straightforward further extension of the (indefinite) stochastic integral discussed in chapter 3, to previsible processes X on which the Daniell-mean $\lceil\!\lceil X \rceil\!\rceil^*_{Z\text{–}2}$ is finite. Actually, in [92] the mean

$$F \mapsto \left(\int^* \left(\int^* |F_s| \, d\,\vdots\widehat{Z}\vdots_s \right)^2 d\mathbb{P} \right)^{1/2} + \left(\int^* \int^* F_s^2 \, d[\widetilde{Z}, \widetilde{Z}]_s \, d\mathbb{P} \right)^{1/2}$$

is used instead, which of course on previsibles is bounded above and below by universal multiples of $\lceil\!\lceil X \rceil\!\rceil^*_{Z\text{–}2}$ ([92, corollary on p. 138]). This second extension is again denoted by $X \cdot Z$ and evidently equals the indefinite integral $X{*}Z$ of the present book.

Now if Z is merely a semimartingale then it is easily seen from the existence of a decomposition $Z = V + M$ that there exist arbitrarily large stopping times T such that Z^{T-}, "the process Z stopped strictly before T" defined as

$$Z^{T-} \stackrel{\text{def}}{=} Z \cdot [\![0, T)\!) + Z_{T-} \cdot [\![T, \infty)\!) \; ,$$

is a global L^2-integrator ([92, theorem 13 on p. 132]). Indeed, for $T \stackrel{\text{def}}{=} \inf\{t : |V_t| \vee |M_t| > n\}$, Z^{T-} has bounded jumps, and corollary 4.4.3 on page 234 shows that this process is actually an L^q-integrator for all q. This suggests to declare a previsible process X (indefinitely) Z-integrable if there exist arbitrarily large[3] stopping times T

such that $\qquad Z^{T-}$ is a global L^2-integrator $\qquad (a)$

and $\qquad X$ is Z^{T-}–2-integrable. $\qquad (b)$

[3] That is to say, for every $\epsilon > 0$ and $t \in (0, \infty)$ there is a stopping time T with $\mathbb{P}[T < t] < \epsilon$ satisfying the condition in question.

Let us call the collection of such stopping times $\mathfrak{T}[Z,X]$. The final extension, the general (indefinite) integral is defined in [92, first definition on p. 134] as the limit of $X \cdot Z^{T-}$, taken as $T \in \mathfrak{T}[Z,X]$ runs through a sequence (T_n) that increases without bound, and is again denoted by $X \mapsto X \cdot Z$.

The existence of a sequence of stopping times T_n satisfying (a) and (b) and increasing almost surely to ∞ can be deduced from the fact that if S,T satisfy $(a),(b)$ then so does $S \vee T$. We show this now. Since, as a little picture will make clear, for $X \in \mathcal{E}$

$$\begin{aligned}\Big|\int X'\,dZ^{(S\vee T)-}\Big| &\le \Big|\int X'\,dZ^{S-}\Big| + \Big|\int X'\,dZ^{T-}\Big| + \Big|X'_S \cdot \Delta Z_S^{T-}\Big| \\ &\le \Big|\int X'\,dZ^{S-}\Big| + \Big|\int X'\,dZ^{T-}\Big| + \Big|\int |X'|^2\,d[Z^{T-},Z^{T-}]\Big|^{1/2},\end{aligned}$$

for any $X' \in \mathcal{E}$ with $|X'| \le |X|$, we have

$$\begin{aligned}\lceil\!\lceil X\rceil\!\rceil^*_{Z^{(S\vee T)-}\text{–}2} &\le \lceil\!\lceil X\rceil\!\rceil^*_{Z^{S-}\text{–}2} + \lceil\!\lceil X\rceil\!\rceil^*_{Z^{T-}\text{–}2} + K_2^{(3.8.6)} \cdot \lceil\!\lceil X\rceil\!\rceil^*_{Z^{T-}\text{–}2} \\ &\le 2\Big(\lceil\!\lceil X\rceil\!\rceil^*_{Z^{S-}\text{–}2} + \lceil\!\lceil X\rceil\!\rceil^*_{Z^{T-}\text{–}2}\Big)\end{aligned}$$

for all $X \in \mathcal{E}$ and then for all $X \in \mathcal{P}$, showing that $S \vee T$ again satisfies both (a) and (b).

The definite stochastic integral $\int_0^t X\,dZ$ is then defined as the value of $X \cdot Z$ at t, at least for finite instants t and previsible integrands X.

There is a little trouble in defining the definite integral $\int_0^\infty X\,dZ$ within this scheme; the definition $\int_0^\infty X\,dZ \overset{\text{def}}{=} \lim_{t\to\infty} \int_0^t X\,dZ$ comes to mind, but this shares the lack of solidity and of the dominated convergence theorem with the improper Riemann integral.

A process X (indefinitely) integrable in the sense of [92] described above is easily seen to be locally Z–0-integrable in the sense of our chapter 3. Namely, since the jump of Z at a $T \in \mathfrak{T}[Z,X]$ is almost surely finite, the sum of the means $F \mapsto \lceil\!\lceil F\rceil\!\rceil^*_{Z^{T-}\text{–}2}$ and $F \mapsto \|F_T \cdot \Delta Z_T\|^*_{L^0}$ is evidently a mean majorizing the Daniell mean $F \mapsto \lceil\!\lceil F\rceil\!\rceil^*_{Z\text{–}0}$ for which X is finite (see definition 3.2.6 on page 97). In view of theorem 3.4.10 and definition 3.7.1, X is Z–0-integrable on the stochastic interval $[\![0,T]\!]$, which expands to $[\![0,\infty)\!)$ as $T \in \mathfrak{T}[Z,X]$ is taken through a sequence increasing without bound. It is clear from exercise 3.7.16 that the indefinite integrals in Daniell's and Protter's sense agree on integrands X as above:

$$X \cdot Z = X{*}Z\ .$$

Therefore the class of processes called integrable in [92] is contained in our class of previsible locally Z–0-integrable processes, with the integrals, both definite and indefinite, being the same at the times they are defined.

These two classes actually agree. To see this we must shoot with a big cannon and invoke the main factorization theorem 4.1.2 on page 191, with $p=2$. Let then $X \in \mathcal{P}$ be Z–0-integrable. Given an arbitrarily large stopping time T there exists a probability $\mathbb{P}'$ equivalent with $\mathbb{P}$ on $\mathcal{F}_T$ so that both Z^T and $X{*}Z^T$ are global $L^2(\mathbb{P}')$-integrators. By equation (3.7.5) on page 135 and theorem 3.4.10 on page 113, the fact that, under $\mathbb{P}'$, $\lceil\!\lceil X\rceil\!\rceil^*_{Z^T\text{–}2} = \lceil X{*}Z^T \rceil_{\mathcal{I}^2}$ is finite implies that X is Z^T–2-integrable under $\mathbb{P}'$. According to [92, theorem 25 on p. 140], X is (indefinitely) Z^T-integrable in the sense of [92] also under $\mathbb{P}$. There exists therefore a stopping time $S \le T$, that can still be had arbitrarily large, so that Z^{S-} is a global $L^2(\mathbb{P})$-integrator and X is Z^{S-}–2-integrable: X meets the definition of [92] of integrability.

It speaks for the ingenuity of the authors who developed the theory without the benefit of hindsight, that the *ad hoc* definition of a semimartingale actually covered all reasonable integrators (i.e., all L^0-integrators) and that the *ad hoc* definition of an (indefinitely) integrable process offered in [74] and slightly gentrified in [92] covered all (at least all previsible) locally Z–0-integrable processes.

I venture a small commercial. Daniell's approach has the appeal of being just a straightforward extension of the usual Lebesgue integral and of straightforwardly extending to random measures — see page 174. In fact, it leads to the definition of a random measure in a somewhat canonical way and thus could be used to unify the disparate definitions and integration theories of random measures that populate the literature.

Erratum at the last line on page 233 The last factor should be 2, not $1/2$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (02/16/2007).

Erratum at Proposition 4.4.7 on page 235 In the last line and in the proof on page 236 replace the subadditive size measurements $\lceil\ \rceil_{\mathcal{I}^p}$ and $[\![\ ]\!]_p$ by their homogeneous versions $|\ |_{\mathcal{I}^p}$ and $\|\ \|_p$, respectively.

Erratum at Exercise 4.4.9 on page 236 With the help of proposition 4.4.7 on page 235 the maps $\zeta \mapsto {}^{\tilde{c}}\zeta$ and $\zeta \mapsto {}^{r}\zeta$ can only be shown to be continuous projections satisfying $|{}^{\tilde{c}}\zeta^{h,t}|_{\mathcal{I}^p} \le C^{p(4.4.2)}|\zeta^{h,t}|_{\mathcal{I}^p}$ and $|{}^{r}\zeta^{h,t}|_{\mathcal{I}^p} \le C^{p(4.4.2)}|\zeta^{h,t}|_{\mathcal{I}^p}$ for $h \in \mathcal{E}_+[\boldsymbol{H}]$ and $t \ge 0$ (see definition 3.10.1 on page 173).
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (02/16/2007).

Erratum at the last line on page 236 This line should read

$$Z'' \stackrel{\text{def}}{=} Z' - M = (1-\Delta)*Z - M\ .$$

Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (02/16/2007).

Erratum at line 19 on page 237 The projections $\boldsymbol{Z} \mapsto {}^{\tilde{c}}\boldsymbol{Z}$ and $\boldsymbol{Z} \mapsto {}^{r}\boldsymbol{Z}$ are only shown to be continuous, not contractive. Also, every Z should be replaced by $\boldsymbol{Z}$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (02/16/2007).

Addendum to line -9 on page 239 In the whole subsection Z continues to be a local L^q-integrator for some $q \in [2,\infty)$. (02/28/2007)

Erratum at line -4 to -3 on page 241 This should read "… the third one follows by taking the p^{th} root after applying Hölder's inequality with conjugate exponents $1/e_\rho$ and $1/e_\tau$ to the p^{th} power of $(*)$".
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (02/16/2007).

Addendum to Lemma 4.5.10 on page 245 Using stopping times that reduce Z to a global L^q-integrator we may clearly assume that Z and with it $¦Z^{[\rho]}¦^{1/\rho}$, $¦\widehat{Z}^{[\rho]}¦^{1/\rho}$, $|\Delta\widehat{Z}|$ etc. are global L^q-integrators.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (02/16/2007).

Erratum at line 2 on page 247 For $\boldsymbol{Z}^{\langle\rho\rangle} = {}^{\rho}\boldsymbol{X} * \boldsymbol{Z}^{\langle\rho\rangle}$ read $\boldsymbol{Z}^{\langle\rho\rangle} = ({}^{\rho}\boldsymbol{X} * \boldsymbol{Z})^{\langle\rho\rangle}$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (02/16/2007).

Erratum at Exercise 4.5.12 on page 247 In the last line of this exercise and of exercises 4.5.13 and 4.5.14 replace $\mathcal{E}^d$ by its sequential closure $(\mathcal{E}^d)^\sigma$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (02/16/2007).

Erratum at line -3 on page 247 For $d\boldsymbol{X}' * \boldsymbol{Z}^{\langle 2\rangle}$ read $d(\boldsymbol{X}' * \boldsymbol{Z})^{\langle 2\rangle}$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (02/16/2007).

Erratum at line 4 on page 249 For ${}^{q}\boldsymbol{X} * \boldsymbol{Z}^{\langle q\rangle}$ read $({}^{q}\boldsymbol{X} * \boldsymbol{Z})^{\langle q\rangle}$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (02/16/2007).

Erratum at line 6 on page 249 For $\mathbb{E}[\,¦{}^{q}\boldsymbol{X} * \boldsymbol{Z}¦_{\infty}^{[q]}]$ read $\mathbb{E}[\,¦({}^{q}\boldsymbol{X} * \boldsymbol{Z})^{[q]}¦_{\infty}]$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (02/16/2007).

Erratum at Exercise 4.5.18 on page 250 The second paragraph is garbled. It should read:

If Z is a continuous local martingale, then $1^\diamond = p^\diamond = 2$ and, up to the factor $C_p^\diamond \le p\sqrt{e/2}$, $\|\ \|_{p\text{–}Z}^\diamond$ agrees with the Hardy mean of definition (4.2.9); thus $\|\ \|_{p\text{–}Z}^\diamond$ is an extension to general integrators of the Hardy mean when $2 \le p < \infty$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (02/16/2007).

Erratum at Equation (4.5.29) on page 250 Inequality (4.5.29) has the factor $\|\mathbf{g}\|_{L^p}$ missing on the right.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (02/16/2007).

Erratum at Exercise 4.5.21 on page 250 In line 3 require $y, a > 0$.
Thanks to **Roger Sewell**, <rfs@cambridgeconsultants.com> (02/16/2007).

Erratum at Exercise 4.5.22 on page 250 In line 3 replace càglàd by càdlàg.
Thanks to **Roger Sewell**, <rfs@cambridgeconsultants.com> (02/16/2007).

Erratum at Exercise 4.5.23 on page 251 Define $A \stackrel{\text{def}}{=} C_q^{\diamond q} \cdot \left(\Lambda^{1/1^\diamond} \vee \Lambda^{1/q^\diamond}\right)$.
Thanks to **Roger Sewell**, <rfs@cambridgeconsultants.com> (02/16/2007).

Erratum at line 7 on page 253 Replace $X'(\eta, \varpi)$ by $X'(\varpi)$.
Thanks to **Roger Sewell**, <rfs@cambridgeconsultants.com> (02/16/2007).

Erratum at line -13 on page 253 Replace cX by $\check{X}$.
Thanks to **Roger Sewell**, <rfs@cambridgeconsultants.com> (02/16/2007).

Addendum to Exercise 4.6.1 on page 254 $\boldsymbol{Z}_{T+.}$ is the map $t \mapsto \boldsymbol{Z}_{T+t}$; T must be finite for $\boldsymbol{Z}_T$ to make sense; and to say $\boldsymbol{Z}'$ is a Lévy process means that it is a Lévy process on its own basic or natural filtration.
Thanks to **Roger Sewell**, <rfs@cambridgeconsultants.com> (03/30/2007).

Erratum at Equation (4.6.4) on page 255 The argument following this inequality is faulty. Please see the web version for a correct version of it.
Thanks to **Roger Sewell**, <rfs@cambridgeconsultants.com> (03/30/2007).

Erratum at line 8 on page 256 For T_n read T^n.
Thanks to **Roger Sewell**, <rfs@cambridgeconsultants.com> (03/30/2007).

Erratum at line -6 on page 257 Replace " is stationary" by "has stationary increments." In line -4 replace $\mathbb{E}[J_1^h]$ by $\mathbb{E}[\sum_{\sigma \le 1} h(\Delta \boldsymbol{Z}_\sigma)]$.
Thanks to **Roger Sewell**, <rfs@cambridgeconsultants.com> (03/30/2007).

Erratum at line 2 ff. on page 259 Replace e_{-s} by e_{s-} in this equation.
Thanks to **Roger Sewell**, <rfs@cambridgeconsultants.com> (03/30/2007).

Erratum at line -5 on page 260 Replace $\int_{[\![0,T]\!]}$ by $\int_{[\![0,t]\!]}$.
Thanks to **Roger Sewell**, <rfs@cambridgeconsultants.com> (03/30/2007).

Erratum at lines 8&9 on page 261 $\boldsymbol{X}$ is $\mathbb{C}^d$-valued, H complex–valued.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (03/30/2007).

Erratum at line 3 on page 261 Delete the spurious reference to lemma 4.6.7.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (03/30/2007).

Erratum at lines 2, 3, 4, 10 on page 266 Put parentheses around $\boldsymbol{X}{*}^{\boldsymbol{\cdot}}\boldsymbol{Z}$ to get $\big(\boldsymbol{X}{*}^{\tilde{s}}\boldsymbol{Z}\big)^\star_t$ etc. Same in inequalities (4.6.28) and (4.6.29) on **page 267**.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (03/30/2007).

Addendum to line 4 on page 266 Replacing $\max_{\rho=2,q}$ by $\max_{\rho=2,p}$ gives a tighter estimate. Same in inequality (4.6.29) on **page 267**.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (03/30/2007).

Erratum at line 2 on page 267 Replace $2^{p-1}C_p$ by $(1+C_p)$. In **line 4** replace “predictable” by “previsible.”
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (03/30/2007).

Erratum at last line on page 268 Replace $\mathbb{E}\big[\phi(\boldsymbol{z}+\boldsymbol{Z}_t)\big]$ by $\mathbb{E}\big[\phi(\boldsymbol{y}+\boldsymbol{Z}_t)\big]$ and $\int_{\mathbb{R}^n}$ by $\int_{\mathbb{R}^d}$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (03/30/2007).

Erratum at line 4 on page 269 Have $\phi \in C_0(\mathbb{R}^d)$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (03/30/2007).

Erratum at the footnote on page 269 For ϕ to be in Schwartz space $\mathcal{S}$ not only ϕ itself but all its partial derivatives must vanish at infinity faster than any power of $1/|x|$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (03/30/2007).

Erratum at line +3 on page 270 For $C_0(\mathbb{R}^n)$ read $C_0(\mathbb{R}^d)$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (05/23/2007).

Erratum at line 16 on page 270 The covariance matrix is tB, not B.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (03/30/2007).

Erratum at line 17 on page 270 Replace $\boldsymbol{A}$ by $\underline{\boldsymbol{A}}$ twice.

Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (03/30/2007).

Erratum at Equation (5.2.5) on page 284 This inequality has the factor $\left\| |\boldsymbol{F}|_\infty \right\|_{p,M}$ missing on the right. It should read

$$\left\|\boldsymbol{F}_{.-}*\boldsymbol{Z}\right\|^{\star}_{p,M} \le \frac{C_p^{\diamond(4.5.1)}}{M^{1/p^\diamond}} \cdot \left\| |\boldsymbol{F}|_\infty \right\|_{p,M} . \tag{D.7}$$

Erratum at Exercise 5.2.3 on page 286 Inequality (5.2.7) should read

$$\left|\left(\boldsymbol{F}[Y]-\boldsymbol{F}[X]\right)\right|_{\infty p} \le L\cdot\left|Y-X\right|_p .$$

Erratum at Exercise 5.2.17 on page 292 Require that ${}^0\boldsymbol{F}$ be Lipschitz but also that ${}^0\boldsymbol{F}[\boldsymbol{X}]$ be bounded at the solution $\boldsymbol{X}$ — see the answer.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (03/07/2007).

Erratum at line +15 on page 313 Replace $\left|X-X^{(n)}\right|^\star_t$ by $\left|X-X^{(n)}\right|^\star_T$.

Addendum to line -8 on page 334 Actually, $\underline{C}^{AB}_\alpha$ is the closure in the supremum norm of the uniformly equicontinuous set C of paths provided by Kolmogoroff's Lemma; as such it is compact. Since $C\subseteq\underline{C}^{AB}_\alpha$, $\mathbb{P}\big[(\underline{X}^\mu_{.}, \underline{\boldsymbol{Z}}^\mu_{.})\in C\big] > 1-\alpha/2$ implies $\mathbb{P}[\Omega^X_\alpha] > 1-\alpha/2$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (08/16/2007).

Erratum at line +5 on page 334 p and M also entered the construction of C_α.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (08/16/2007).

Erratum at line -6 on page 335 Insert $\boldsymbol{f}$ before $[\boldsymbol{Z}', X']$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (08/16/2007).

Erratum at Equation (5.5.12) on page 336 Replace $\boldsymbol{X}$ by $\boldsymbol{X}^{(n)}$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (08/16/2007).

Erratum at Theorem 5.6.1 on page 344 The original proof of the surjectivity (ii) was wrong; I rewrote the whole subsection to correct it.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (02/14/2009).

Erratum at line +15 on page 369 In the definition of $\overline{Z}$ replace $\forall\,\phi \in \mathcal{A}$ by $\forall\,\widehat{\phi} \in \mathcal{A}$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (08/31/2005).

Addendum to Exercise A.2.3 on page 369 $\overline{\rho}(f_1, f_2)$ is the function on $\boldsymbol{B}$ that takes $b \in \boldsymbol{B}$ to $\overline{\rho}\big(f_1(b), f_2(b)\big)$. Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (08/31/2005).

Addendum to Lemma A.2.16 on page 375 , part (i) In other (Roger Sewell's clearer) words: The uniformity generated by $\mathcal{E}$ coincides with the uniformity generated by the smallest uniformly closed algebra containing $\mathcal{E}$ and the constants.

Erratum at line 3 of the proof on page 378 γ^K is lower semicontinuous.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (10/09/2006).

Erratum at line -3 on page 378 Delete a spurious $\{$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (10/09/2006).

Erratum at Theorem A.2.25 on page 379 The sets K and C must be **disjoint**; without this assumption the statement is obviously false.
Thanks to **Pedro Fortuny** `<pfortuny@sdf-eu.org>` (06/05/2004).
More is wrong with the statement: $\mathcal{V}$ must be locally convex, and one must admit the possibility that $x^*(k) > 1$ for all $k \in K$ and $x^*(x) \le 1$ for all $x \in C$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (09/09/2006).
It is best to restate an enhanced version of the theorem:

Theorem D.6 *Let $\mathcal{V}$ be a locally convex topological vector space.*
(i) Let $A, B \subset \mathcal{V}$ be convex, non–void, and disjoint, A closed and B either open or compact. There exist a continuous linear functional $x^* : \mathcal{V} \to \mathbb{R}$ and a number c so that $x^*(a) \le c$ for all $a \in A$ and $x^*(b) > c$ for all $b \in B$.
(ii) **(Hahn-Banach)** *A linear functional defined and continuous on a linear subspace of $\mathcal{V}$ has an extension to a continuous linear functional on all of $\mathcal{V}$.*
(iii) A convex subset of $\mathcal{V}$ is closed if and only if it is weakly closed.
(iv) **(Alaoglu)** *An equicontinuous set of linear functionals on $\mathcal{V}$ is relatively weak*–compact.*

For a proof see appendix C, *Answers to Most Problems*,
at `www.ma.utexas.edu/users/kbi/SDE/C_1AxA.pdf`.

Erratum at line -6 on page 380 The Minkowski functional is defined by $\| f \| \stackrel{\text{def}}{=} \inf\{|r| : f/r \in V\}$ instead of $\| f \| \stackrel{\text{def}}{=} \inf\{|r| : rf \in V\}$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (09/09/2005).

Erratum at line -5 on page 382 As $\eta < 0$, replace $(-\eta/2, 0)$ by $(\eta/2, 0)$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (09/09/2006).

Erratum at line 20 on page 383 Replace K' by K: "$\ldots h'$ is strictly negative on all of $K \ldots$"
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (02/16/2007).

Erratum at line -11 on page 391 The claim that the limit of a sequence of Borel maps is Borel is false, and the sketch of a proof given is embarassing. Thanks to **Oliver Diaz–Espinoza**, `odiaz@math.mcmaster.ca` for pointing out the following counterexample from page 96 of [27]. Let $f = I$ be the unit interval, and equip $G = I^I$ with the topology of pointwise convergence. For every $x \in I$ let $f_n(x) \in I^I$ be the function $y \mapsto max(0, 1 - n|x - y|)$. The maps $f_n : I \to I^I$ are continuous, but their pointwise limit f, which maps every $x \in I$ to $1_{\{x\}} : y \mapsto [x = y]$ is not Borel measurable: for a non–measurable $B \subset I$ the set $U \stackrel{\text{def}}{=} \{\xi \in I^I : \exists x \in B \text{ with } \xi(x) > 0\}$ is open in I^I, yet $f^{-1}(U) = B$. (09/04/2006)

Erratum at Theorem A.3.24 on page 408 In *(iii)* "Then if $\mu(1) = 1 \ \ldots$" must be substituted for "Then if $\mu(1) \geq 1 \ \ldots$"
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (09/14/2206).

Erratum at line -12 on page 416 Replace $(E, \mathcal{A}, \mu)$ by $(F, \mathcal{A}, \mu)$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (02/16/2007).

Erratum at line -4 on page 416 Lower the subscript σ: $\mathcal{A} \stackrel{\text{def}}{=} \bigcup_{\sigma \in \Sigma} \mathcal{A}_\sigma$ and remove a spurious "and."
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (02/16/2007).

Erratum at line 12 on page 417 Read $T^{\mathcal{B}} A = T^{\mathcal{A}_n} A : \ldots$
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (02/16/2007).

Erratum at lines 14–24 on page 417 To avoid confusion with the ambient set, whose name is also F, replace every occurrence of F in the second paragraph by G.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (02/16/2007).

Erratum at line 17 on page 418 Replace $[X_i > 1 - 1/i]$ by $[X_i > 1/i]$.

Erratum at end of line 24 on page 418 Remove a spurious "in."
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (02/16/2007).

Erratum at line -8 on page 418 Require $\sum a_i \mu^{K_i}(P_i) < \infty$ instead.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (02/16/2007).

Erratum at lines 5–8 on page 419 Replace X_k by Y_k.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (02/16/2007).

Erratum at line +4 on page 430 For "conjuction" read "conjunction."

Erratum at line +6 on page 433 The definition of the B'_n should read

$$B'_n = \prod_{m \neq n} K_m \times B_n = \prod_{m \neq n} K_m \times \bigcap_{j=1}^{\infty} B_n^j \subset F \times K .$$

Thanks to **Roger Sewell** `<rfs@cambridgeconsultants.com>` (07/08/2005).

Erratum at Exercise A.5.6 on page 434 Omit the spurious parenthetical "(but not necessarily closed)" in part (iii) and replace $\mathcal{K}^{\cap_a \cup_f}$ by $\mathcal{K}^{\cup_f \cap_a}$ in the answer.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (12/21/2006).

Erratum at Lemma A.5.8 on page 434 The proof of lemma A.5.8 (ii) requires the assumption that $\mathcal{F}$ be closed under finite unions. Also, the sequence (C_n) in (i) must be decreasing. [This is satisfied in all subsequent applications of this lemma (Theorems A.5.9 and A.5.10).] Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (07/18/2005 and 12/21/2006).

Erratum at line +22 on page 435 Replace "$\mathcal{F}_{\sigma\delta}$ have…" by "$\mathcal{F}$ have…"
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (12/21/2006).

Erratum at line -1 on page 439 Replace $\sum_k \overline{M}_{k2^{-n}}\big[k2^{-n},(k+1)2^{-n}\big)$ by $\sum_k \overline{M}^g_{k2^{-n}}\big(k2^{-n},(k+1)2^{-n}\big]$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (12/21/2006).

Erratum at line +15 on page 448 For "bsince" read "since."
Thanks to **Mohamoud Dualeh** `<mabaduuk@yahoo.com>` (10/22/2003).

Erratum at line -13 on page 451 For "subset B" read "subset $\mathcal{C}$."

Erratum at line +2 on page 459 The term $e^{-\alpha^q}$ in the very first integral should be replaced by $e^{-|\alpha|^q}$.

Erratum at line 1 of the Proof on page 460 The line should read
(i) The functions $f \stackrel{\text{def}}{=} \sum_\nu c_\nu \gamma_\nu^{(q)}$ and ...
so that the function f appearing in the proof of (ii) is now defined.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (10/09/2006).

Erratum at line -5 on page 463 Read $\alpha U_\alpha \phi \xrightarrow[\alpha\to\infty]{} \phi$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (03/30/2007).

Erratum at line -10 on page 464 The reference should be to (A.9.4).
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (03/30/2007).

Erratum at last line on page 466 Replace $\{\mu_t : t > 0\}$ by $\{\mu_t : t \geq 0\}$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (03/30/2007).
