Pseudodifferential Operators

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The study of pseudodifferential operators emerged in the 1960's, having its origins in the study of singular integro-differential operators. In fact, Friedrichs and Lax coined the term "pseudodifferential operator" in their 1965 paper entitled "Boundary Value Problems for First Order Operators". Since that time, pseudodifferential operators have proved useful in many arenas of modern analysis and mathematical physics. They are particularly important to the study of elliptic equations and in the index theory for elliptic operators. Pseudodifferential operators "allow us not only to establish new theorems but also to have a fresh look at old ones and thereby obtain simpler and more transparent formulations of already known facts [3]."

The aim of this paper will be to give an overview of pseudodifferential operators. We will motivate and define them and develop several of their important properties. Then we will define elliptic pseudodifferential operators and give several applications of them. In large part, this paper will follow the treatment in [2], with the most notable digression concerning the spectrum of elliptic operators.

1 Preliminaries

Before describing pseudodifferential operators, we would like to introduce the notions of local and pseudolocal operators. Also, we will recall the definitions of both the Fourier transform and Sobolev spaces. We use the standard notation $\mathcal{D}'(\cdot)$ and $\mathcal{E}'(\cdot)$ to denote the dual spaces of $C_c^\infty(\cdot)$ and $C^\infty(\cdot)$, respectively.

Definition 1. *Let M be a smooth, differentiable manifold.*

1. *If $L : C_0^\infty(M) \to C^\infty(M)$ is linear, we say that L is a* local operator *if supp Lu is contained in supp u for every u in $C_0^\infty(M)$.*

2. *If $L : \mathcal{E}'(M) \to \mathcal{D}'(M)$ is linear, we say that L is a* pseudolocal operator *if sing supp Lu is contained in sing supp u for every u in $\mathcal{E}'(M)$. Recall that the singular support of T is the smallest closed set in the complement of which T, as a distribution, is a C^∞ function.*

It is a well-known fact that every differential operator, i.e. a polynomial in $D = (D_1, \ldots, D_n)$ with coefficients in C^∞, is local and that, in fact, every local operator is a differential operator. Additionally, every differential operator is pseudolocal because if $p(x, D)$ is a differential operator and if T is a C^∞ function in an open set O, then so also is $p(x, D)T$.

We would also like to describe the Fourier transform. Let $\mathcal{S}$ be the Schwartz space of C^∞ functions that decay rapidly at ∞. Specifically, if $u \in \mathcal{S}$, then

$$\sup_{x\in\mathbf{R^n}} ((1+|x|)^M \sum_{|\alpha|\le m} |\partial^\alpha u(x)|) < \infty,$$

for any nonnegative integers m and M. For such a u, we can define its Fourier transform $\hat{u}$ by the following formula:

$$\hat{u}(\xi) = (2\pi)^{-n} \int e^{-ix\cdot\xi} u(x)dx. \tag{1}$$

The inverse Fourier transform is given by

$$u(x) = \int e^{i\xi\cdot x}\hat{u}(\xi)d\xi. \tag{2}$$

The Plancherel theorem implies $\|u\|_{L_2} = \|\hat{u}\|_{L_2}$, while a standard identity gives that $D\hat{^\alpha} u(\xi) = \xi^\alpha \hat{u}(\xi)$.

Finally, we will define Sobolev spaces and give several useful characterizations. For any real number s, let

$$\|u\|_s = (\int |\hat{u}(\xi)|^2(1+|\xi|^2)^s d\xi)^{\frac{1}{2}}$$

hold for any $u \in C_0^\infty(\mathbf{R^n})$. We define the Sobolev space $H_s(\mathbf{R^n})$ to be the completion of C_0^∞ under this norm. Notice that we can define $H_s(M)$ for M a compact manifold without boundary by using a finite covering $\{U_i\}$ of M by coordinate patches and a subordinate partition of unity ρ_i. Then

$$H_s(M) = \{u \in \mathcal{D}'(M) : \rho_i u \in H_s(\mathbf{R^n})\text{for all i }\},$$

and

$$\|u\|_s = (\sum_i \|\rho_i u\|_s^2)^{\frac{1}{2}}.$$

For $s > r$, $H_s(M)$ is compactly contained in $H_r(M)$, i.e. bounded sets in $H_s(M)$ are precompact in $H_r(M)$.

2 Basic Properties of Pseudodifferential Operators

In this section, we will motivate and define pseudodifferential operators. Several key attributes will also be stated and proved.

To motivate the definition of a pseudodifferential operator, consider a differential operator $p(x,D)$ of the form $p(x,D) = \sum_{|\alpha|\le m} c_\alpha(x)D^\alpha$. From equation 2, we have

$$\begin{aligned} p(x,D)u(x) &= \sum c_\alpha(x) \int e^{ix\cdot\xi}\xi^\alpha \hat{u}(\xi)d\xi \\ &= \int e^{ix\cdot\xi} \sum c_\alpha(x)\xi^\alpha \hat{u}(\xi)d\xi \\ &= \int e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)d\xi. \end{aligned}$$

In this notation, the polynomial $p(x,\xi)$ is called the *symbol* of $p(x,D)$. We would like to make an analogous definition, but for more general symbols. As such, we will define a *pseudodifferential operator* P using the formula

$$Pu(x) = \int e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)d\xi. \tag{3}$$

Using the definition of the Fourier transform, the following alternate form of P arises:

$$\begin{aligned} Pu(x) &= \int\int e^{i\xi\cdot(x-z)} p(x,\xi)u(z)dzd\xi & (4)\\ &= \int K(x,x-y)u(y)dy, & (5) \end{aligned}$$

where

$$K(x,z) = (2\pi)^{-n}\int e^{i\xi\cdot x} p(x,\xi)d\xi. \tag{6}$$

We want to ensure that a pseudodifferential operator is pseudolocal. Unfortunately, not every choice of $p(x,\xi)$ will yield this property. For example, if $p(x,\xi) = e^{i\xi\cdot x_0}$, for x_0 fixed, then the corresponding P is clearly not pseudolocal, as it acts as a shift operator. Thus $p(x,\xi)$ cannot be an arbitrary function; instead we will require that p be smooth and that high ξ derivatives of p rapidly decay at ∞.

Definition 2. *A function $p \in C^\infty(\mathbf{R^n}\times\mathbf{R^n})$ is called a* symbol of order m *if for every compact set K in $\mathbf{R^n}$ and for every α,β there exists $c = c(\alpha,\beta,K)$ such that, for all $(x,\xi)\in K\times\mathbf{R^n}$,*

$$|\partial_\xi^\alpha D_x^\beta p(x,\xi)| \leq c(1+|\xi|)^{m-|\alpha|}. \tag{7}$$

In this case, we write $p \in S^m$.

Theorem 2.1. *If p is in S^m, then equation 3 defines a linear operator $P : C_0^\infty(\mathbf{R^n}) \to C^\infty(\mathbf{R^n})$.*

Proof. From the definition, P is clearly linear. Suppose $u \in C_0^\infty(\mathbf{R^n})$. Since $C_0^\infty \subset \mathcal{S}$, we have that $\hat{u} \in \mathcal{S}$, so $\hat{u}$ decays rapidly at ∞. For any integer N and any multi-index β, we have

$$|D_x^\beta p(x,\xi)\hat{u}(\xi)| \leq c(1+|\xi|)^m(1+|\xi|)^{-N}.$$

Thus differentiation under the integral sign is justified, and $Pu \in C^\infty$ as required. □

Let M be a compact manifold. Suppose L is a linear operator such that $L : C_0^\infty(M) \to C^\infty(M)$ extends to a continuous operator $L : H_{s+m}(M) \to H_s(M)$ for all real s. Such an L has *order m*. Of particular interest is the set of operators of order $-\infty$, denoted $\mathcal{L}_{-\infty}$. Since $H_s(M) \subset H_r(M)$ for all $s \geq r$, and since $\bigcap_s H_s(M) \subset C^\infty(M)$, we see that $L(H_s(M)) \subset C^\infty(M)$, for $L \in \mathcal{L}_{-\infty}$. These operators are sometimes called regularizing [4]. We will ultimately consider pseudodifferential operators modulo the $\mathcal{L}_{-\infty}$ operators.

Theorem 2.2. *If p is in S^m, then P has order m [2].*

Corollary 2.3. *Every pseudodifferential operator P defines a continuous linear mapping of C_0^∞ into C^∞ which extends as a continuous linear mapping of $\mathcal{E}'$ into $\mathcal{D}'$ [4].*

Consider an operator of the form $Au(x) = \int K(x,y)u(y)dy$. $K(x,y)$ is said to be *separately regular in x and y* if A maps C_0^∞ into C^∞ and extends to a map of $\mathcal{E}'$ into $\mathcal{D}'$. Additionally, $K(x,y)$ is *very regular* if it is separately regular and is a C^∞ function off the diagonal [4].

Lemma 2.4. *If $K(x,y)$ is very regular, the associated operator A is pseudolocal.*

Proof. We want to show that sing supp Au is contained in sing supp u for any $u \in \mathcal{E}'$. Let U and V be open sets with $V \subset\subset U$. By the characterization of $\mathcal{E}'$, we can find u such that $u \in C^\infty(U)$. Also let $g \in C_0^\infty(U)$, where $g \equiv 1$ in a neighborhood of $\overline{V}$. Write $u = gu + (1-g)u$, so that $Au = A(gu) + A((1-g)u)$. Since $K(x,y)$ is separately regular, $gu \in C_0^\infty$ implies $A(gu) \in C^\infty$. Additionally,

$$A((1-g)u) = \int K(x,y)(1-g(y))u(y)dy,$$

so if $x \in V$ and $y \in$ supp $1-g$, then (x,y) is a suitable distance from the diagonal. Thus $K(x,y)$ is a C^∞ function. Differentiating under the integral sign shows that $K((1-g)u)$ is C^∞ on V. Thus A is pseudolocal. □

We would like to show that pseudodifferential operators are, in fact, pseudolocal. In light of lemma 2.4, it remains only to show that $K(x,y)$ as defined by equation 6 is very regular. Corollary 2.3 implies $K(x,y)$ is separately regular, and so it remains to show that K is a C^∞ function off the diagonal. Notice that repeated integration by parts gives, for any positive integer N,

$$K(x,y) = (i)^N \int \frac{e^{i\xi\cdot x}}{y_1^N}(\frac{\partial}{\partial \xi_1})^N p(x,\xi)d\xi.$$

Recalling that $p \in S^m$, we have the following estimate:

$$|\partial_x^\beta \partial_y^\gamma K(x,y)| \le C \int (|1+|\xi|^{|\gamma|})(|y_1|^{-N} + |y_1|^{-N-|\gamma|})|\partial_x^\beta(\frac{\partial}{\partial \xi_1})^N p(x,\xi)d\xi.$$

This estimate shows that we can differentiate under the integral sign in equation 6 and thus that $K(x,y)$ is C^∞ off the diagonal. Thus we have proved the following theorem:

Theorem 2.5. *A pseudodifferential operator P is pseudolocal.*

3 Elliptic Pseudodifferential Operators

In this section, we will define elliptic pseudodifferential operators and show that they are invertible modulo $\mathcal{L}_{-\infty}$. A specific use of elliptic pseudodifferential operators will also be illustrated.

Definition 3. *Let P be a pseudodifferential operator with symbol $p \in S^m$ a square matrix. $p(x,\xi)$ is said to be* elliptic *if it is invertible for large $|\xi|$, and if there exists a constant C such that*

$$|p(x,\xi)^{-1}| \le C(1+|\xi|)^{-m}, \tag{8}$$

for $|\xi| \ge C$. An elliptic pseudodifferential operator *will be one with an elliptic symbol.*

The pseudodifferential operators are closed under composition. In order to define this operation, we would like to introduce the following notation. Let $\{q_j\}$ be a collection of symbols such that $q_j \in S^{m_j}$, where the $m_j \to -\infty$. Then there exists a $p \in S^{m_0}$ such that $p - \sum_{j=0}^{N-1} q_j \in S^{m_N}$. Such a p will be denoted $p \sim \sum q_j$.

Theorem 3.1. *Let P and Q be pseudodifferential operators with symbols $p \in S^m$ and $q \in S^n$. Then $R = QP$ is a pseudodifferential operator with symbol $r(x,\xi) \in S^{m+n}$, and*

$$r(x,\xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_\xi^\alpha q(x,\xi) D_x^\alpha p(x,\xi). \tag{9}$$

Proof. Using the definitions of P and Q,

$$\begin{aligned} Ru(x) &= QPu(x) \\ &= (2\pi)^{-n} \int\int e^{i\xi\cdot(x-y)} q(x,\xi) \int e^{iy\cdot\eta} p(y,\eta)\hat{u}(\eta) d\eta dy d\xi \\ &= \int r(x,\eta) e^{i\eta\cdot x} \hat{u}(\eta) d\eta. \end{aligned}$$

Recall the Taylor expansion for a function $u(x,z)$:

$$u(x,z) = \sum_{|\alpha|\le N} \left(\frac{1}{\alpha!} \partial^\alpha u(x,z)(y-z)^\alpha\right) + r_N,$$

where r_N is the remainder. We can apply this expansion to $r(x,\eta)$ and obtain

$$\begin{aligned} r(x,\eta) &= (2\pi)^{-n} \int\int q(x,\xi) p(y,\eta) e^{i(\xi-\eta)\cdot(x-y)} dy d\xi \\ &= (2\pi)^{-n} \int\int \Big(\sum_{|\alpha|\le N} \frac{1}{\alpha!} \partial_\eta^\alpha q(x,\eta)(\xi-\eta)^\alpha\Big) \cdot p(y,\eta) e^{i(\xi-\eta)\cdot(x-y)} dy d\xi + r_N. \end{aligned}$$

Thus one sees that

$$r(x,\eta) = \sum_{|\alpha|\le N} \frac{1}{\alpha!} \partial_\eta^\alpha q(x,\eta) D_x^\alpha p(x,\eta) + r_N,$$

as required. ∎

Using this definition of composition, we will show that elliptic pseudodifferential operators are invertible modulo $\mathcal{L}_{-\infty}$. Thus an elliptic pseudodifferential operator, P, has the following property:

> Given any distribution $u \in \mathcal{E}'$ and any open subset U, if $Pu \in C^\infty(U)$, then $u \in C^\infty(U)$. (Nirenberg refers to this property as *hypoellipticity* [2]).

Notice that if a pseudodifferential operator, P, has a left inverse modulo $\mathcal{L}_{-\infty}$, Q, then $u = QPu + Tu$. $Tu \in C^\infty$ because T is regularizing. Since Q is pseudolocal and $Pu \in C^\infty$, so is QPu.

Theorem 3.2. *Let P be an elliptic pseudodifferential operator of order m. There exists a Q with order $-m$ such that $PQ \equiv QP \equiv I$ mod $\mathcal{L}_{-\infty}$.*

Proof. Construct the left inverse Q, with symbol $q \in S^{-m}$, using the sequence $\{q_j\}$ such that $q_j \in S^{-m-j}$ and $q \sim \sum q_j$. Equation 9 yields the following recursively defined formulas:

$$q_0(x,\xi)p(x,\xi) = 1,$$

and for $j = 1, 2, \ldots$,

$$q_j(x,\xi)p(x,\xi) = -\sum_{1\le|\alpha|\le j} \partial_\xi^x q_{j-|\alpha|}(x,\xi) D_x^\alpha p(x,\xi).$$

Then Q will be of order $-m$, and it will be a left inverse of P. We could construct a right inverse, W for P using a similar construction; however, notice that $Q = QPW = W$ mod $\mathcal{L}_{-\infty}$. Thus Q is both a left and right inverse for P. ☐

Index: An application of elliptic pseudodifferential operators

We would like to present one use of elliptic pseudodifferential operators via the calculation of indices. This result is taken from the 1960 paper by R.T. Seeley entitled "Regularization of Singular Integral Operators on Compact Manifolds".

Definition 4. *Let L be a bounded linear map between two Banach spaces X and Y.*

1. *L is a* compact operator *if the image under L of any bounded set in X is precompact in Y.*

2. *L is a* Fredholm operator *if the dimension of its null space and the codimension of its range are both finite.*

3. *If L is a Fredholm operator, define the* index *of L to be*

$$ind\ L = dim\ N(L) - codim\ R(L) = dim\ N(L) - dim\ N(L^*).$$

Theorem 3.3. *With the same hypotheses as above, L is a Fredholm operator if and only if there exists a bounded linear map $B : Y \to X$ such that $LB - I_Y$ and $BL - I_X$ are compact operators.*

Using these definitions and theorem, it is easy to see, using the Sobolev embedding theorem and the definition of the order of a pseudodifferential operator, that if $R \in \mathcal{L}_{-\infty}$, then R is a compact operator. Thus, for P an elliptic pseudodifferential operator of order m, theorems 3.2 and 3.3 imply that $P : H_s(M) \to H_{s-m}(M)$ is a Fredholm operator for M a compact manifold and s any real number. So we can speak of the index of an elliptic pseudodifferential operator.

The following result is that of Seeley; we state it as in [2] without proof.

Theorem 3.4. *Let P be an elliptic pseudodifferential operator acting on complex valued functions on a compact manifold M of dimension > 2. Then for each s, $P : H_s(M) \to H_{s-m}(M)$ has index 0.*

4 Spectral Theory for Elliptic Pseudodifferential and Differential Operators

Finally, we would like to give a formal asymptotic result for the behavior of the the eigenvalues of elliptic differential operators. Following the treatment in [3], we will give a more precise formulation of the conditions on hypoellipticity and will provide several important properties of the spectrum.

Definition 5. *A symbol $p \in C^\infty(U \times \mathbf{R^n})$ is said to be* hypoelliptic *if the following hold:*

i. *There exist real numbers m_0 and m such that if K is some compact subset of X and $|\xi| \geq R$ for some positive R, then*

$$C_1|\xi|^{m_0} \leq |p(x,\xi)| \leq C_2|\xi|^m,$$

holds for constants C_1 and C_2.

ii. *There exists a and b, with $0 \leq a < b \leq 1$, and for any compact $K \subset X$ a constant R such that for any multi-indices α and β,*

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi) p^{-1}(x,\xi)| \leq C(\alpha,\beta,K)|\xi|^{-b|\alpha|+a|\beta|},$$

holds for $|\xi| \geq R$ and $x \in K$.

The class of properly supported pseudodifferential operators P having symbol p as above will be denoted $HL^{m_0,m}_{a,b}$, and we will call P *hypoelliptic* if there exists $P_1 \in HL^{m_0,m}_{a,b}$ and $L \in \mathcal{L}_{-\infty}$ such that $P = P_1 + L$. Notice then that P is elliptic if $P \in HL^{m,m}_{a,b}$.

Suppose X is a Banach space and that $T : X \to X$ is a bounded linear operator. Let T_λ denote the operator $T - \lambda I_X$, where λ is a complex number. The collection of all λ such that T_λ is not invertible is defined to be the *spectrum* of T, denoted $\sigma(T)$. If T_λ is invertible, then λ is in the *resolvent set*, and T_λ^{-1} is called the *resolvent operator* [1].

The following theorem is a well-known result for self-adjoint bounded linear operators stated in the language of elliptic pseudodifferential operators. (Note that the formal adjoint of a pseudodifferential operator can be constructed using ideas similar to that of theorem 9. For details, see [2] or [4].)

Theorem 4.1. *Let $P \in HL^{m,m}_{a,b}(M)$, $1-b \leq a \leq b$, be self-adjoint. Then for $\phi_i \in C^\infty(M)$, all λ_i such that $P\phi_i = \lambda_i\phi_i$ are real, and $|\lambda_i| \to \infty$ and $i \to \infty$. Also $\sigma(P) = \{\lambda_i\}$.*

We would now like to formally describe the asymptotic behavior of the eigenvalues of an elliptic differential operator. Shubin defines an *elliptic differential operator* to be one such that the principal symbol $a_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ is never zero. Using the notation above, this is equivalent to saying $A \in HL^{m,m}_{1,0}(M)$ [3].

Let A be a positive self-adjoint elliptic differential operator on a compact manifold M having eigenvalues $\{\lambda_i\}$. Suppose the λ_i are such that $0 < \lambda_1 \leq \lambda_2 \leq \ldots$. Define the function $N(t) = \sum_{\lambda_i \leq t} 1$, so that $N(t)$ counts, with multiplicity, the number of eigenvalues of A less than or equal to t. Also define the function $V(t)$, which will be the volume in T^*M. We state the following theorem as in [3]:

Theorem 4.2. *We have the following asymptotic formula:*

$$N(t) \sim V(t). \tag{10}$$

The proof of this theorem is quite long and complicated, and it is not immediately clear that it has any application to our study. However, Shubin uses pseudodifferential operators and the closely related Fourier Integral Operators to describe such functions of elliptic differential operators as the complex powers, exponents, and approximate spectral projections; with these characterizations, he can acheive the above result. For a detailed reference, see [3].

5 Conclusion

We have defined and described pseudodifferential operators with a particular look at the subclass of elliptic pseudodifferential operators. Additionally, we have seen how these operators can be used to give results about differential operators. Although this paper has been but a brief overview, we hope the reader has gained some insight into this rich subject.

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