Final Exam
M365C: Analysis I
December 8, 2016

T/F. _____/30 1. _____/20 2. _____/10 3. _____/10 4. _____/15
5. _____/15  Total Score. _____/100
6 (EC). _____/10  7 (EC). _____/10  8 (EC). _____/10

Please put your name at the top of the exam. Read over the entire exam before you begin. Continue your work on the backs of pages or on extra sheets. If your solution runs over onto these pages, please indicate that clearly. If you use extra sheets, be sure to staple them to the rest of your exam.

Only attempt the extra credit problems after completing the rest of the exam. Extra credit problems are scored separately and are not included in the Total Score.

Write neatly! The more neat work you show, the more (partial) credit you will receive.
Have fun!

Answer the following true/false questions. If you find a question ambiguous, write a brief explanation. Each true/false question is worth 2 points.

True  False

☐ 1. Fix real numbers $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a monotone function. Then $f$ is Riemann integrable.

☐ 2. If $x \in \mathbb{R}$ satisfies $-\epsilon < x < \epsilon$ for all $\epsilon > 0$, then $x = 0$.

☐ 3. Let $s_{m,n} = \frac{m}{m+n}$ for $n, m = 1, 2, 3, \ldots$ Then

\[
\lim_{n \to \infty} \lim_{m \to \infty} s_{m,n} = \lim_{m \to \infty} \lim_{n \to \infty} s_{m,n}.
\]

☐ 4. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Then $f(x) = 0$ for all $x \in \mathbb{R}$.

☐ 5. Fix real numbers $a < b$, and suppose $f_1, f_2 : [a, b] \to \mathbb{R}$ are Riemann integrable. If $f_1(x) \leq f_2(x)$ for all $x \in [a, b]$, then

\[
\int_{a}^{b} f_1 \, dx \leq \int_{a}^{b} f_2 \, dx.
\]
6. If \( f_n : (0, 1) \to \mathbb{R} \) is a sequence of differentiable functions, and for each \( x \in (0, 1) \) we have \( f_n(x) \to f(x) \) for a function \( f : (0, 1) \to \mathbb{R} \), then \( f \) is differentiable.

7. Let \( f : [0, 1] \to \mathbb{R} \) be monotonic increasing. Then \( f \) is necessarily continuous.

8. Let \( X \) be a metric space and \( E \subset X \) a compact subset. Then \( \overline{E} = E \).

9. The function \( f : [0, 1] \to \mathbb{R} \) defined by

\[
f(x) = \begin{cases} 
1/q, & x = p/q, \quad p, q \geq 0 \text{ relatively prime;} \\
0, & x \text{ irrational}
\end{cases}
\]

is Riemann integrable and \( \int_0^1 f \, dx = 0 \).

10. Let \( (X, d) \) be a metric space and \( \{K_\alpha\}_{\alpha \in A} \) a collection of compact subsets of \( X \) indexed by a set \( A \). Then \( \bigcup_{\alpha \in A} K_\alpha \) is compact.

11. There exists an unbounded continuous function \( f : [0, 1] \to \mathbb{R} \).

12. If \( \{x_n\} \subset \mathbb{Q} \) is a Cauchy sequence, then there exists \( x \in \mathbb{Q} \) such that \( x_n \to x \).

13. The function \( f : [0, 1] \to \mathbb{R} \) defined by the formula

\[
f(x) = \int_0^x \sqrt{t} \, dt
\]

is differentiable at \( x = 1/2 \) and \( f'(1/2) = 1/\sqrt{2} \).

14. If \( f : (a, b) \to \mathbb{R} \) is a differentiable function defined on an interval \( (a, b) \subset \mathbb{R} \), and \( f'(x) > 0 \) for all \( x \in (a, b) \), then there is an inverse function \( g \) and

\[
g'(f(x)) = \frac{1}{f'(x)}
\]

for all \( x \in (a, b) \).

15. \( \sum_{n=1}^\infty \frac{1}{1 + n^2 x} \) converges uniformly for all \( x \in (-1, 1) \).
1. (20 points)

Displayed are the graphs of five functions $f_i: (-1, 1) \to \mathbb{R}$, $i = 1, 2, 3, 4, 5$. (See the next page for a larger image.) For each graph, tell for which $i$ it is the graph of $f_i$ (1–5), if $f_i$ is bounded, uniformly continuous, differentiable, and whether $\lim_{\epsilon \to 0} \int_{-1+\epsilon}^{1-\epsilon} f_i \, dx$ is defined and exists. The last question is Integrable?. The entry in the first column is a number 1–5; in the remaining columns Y or N.

$$f_1(x) = x^3 \quad f_2(x) = x^4 \quad f_3(x) = \frac{x^4}{1 - x^2}$$

$$f_4(x) = \begin{cases} 
\frac{1}{x}, & x \neq 0; \\
0, & x = 0 
\end{cases} \quad f_5(x) = 1 - |x|$$

<table>
<thead>
<tr>
<th>$f_i$</th>
<th>Bounded?</th>
<th>Uniformly Continuous?</th>
<th>Differentiable?</th>
<th>Integrable?</th>
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2. (10 points) Consider the function \( f: \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = x|x| \).

(a) Show that \( f \) is continuously differentiable, i.e., \( f': \mathbb{R} \to \mathbb{R} \) exists and is continuous.

(b) Show that \( f''(0) \) does not exist.

\[
\begin{align*}
    f(x) &= \begin{cases} 
        x^2, & x > 0 \\
        -x^2, & x < 0 
    \end{cases} \\
    f'(x) &= \begin{cases} 
        2x, & x > 0 \\
        -2x, & x < 0 
    \end{cases} \\
    f''(x) &= \begin{cases} 
        2, & x > 0 \\
        -2, & x < 0 
    \end{cases}
\end{align*}
\]

is continuous. You need to compute \( f'(0) \) as the limit

\[
f'(0) = \lim_{t \to 0} \frac{f(t) - f(0)}{t} = \lim_{t \to 0} \frac{|t|}{t} = 0.
\]

Then \( f' \) is continuous.

(b) The difference quotient which defines \( f'(0) \) is \( \frac{f(x) - f(0)}{x} \) and equals

\[
\begin{align*}
    \frac{f'(x) - f'(0)}{x} &= \begin{cases} 
        \frac{2x}{x}, & x > 0 \\
        \frac{-2x}{x}, & x < 0 
    \end{cases} = \begin{cases} 
        2, & x > 0 \\
        -2, & x < 0 
    \end{cases}
\end{align*}
\]

There is no limit as \( x \to 0 \).
3. (10 points) For each of the pairs of metric spaces $X,Y$ listed below, either describe (through formulas, pictures, and/or words) a surjective (onto) continuous function $f: X \to Y$ or explain why none exists. All of the spaces are subspaces of $\mathbb{R}$ with the standard metric, each of which is an open interval, closed interval, a union of closed intervals, or a set consisting of two points.

(a) $X = (0,1)$ \quad $Y = \{0,1\}$
(b) $X = [0,1]$ \quad $Y = (0,1)$
(c) $X = [0,1] \cup [2,3]$ \quad $Y = \{0,1\}$
(d) $X = (0,1)$ \quad $Y = [0,1]$

(a) None; the continuous image of a connected set is connected.

(b) \begin{center} compact --- compact \end{center}

(c) \begin{equation*} f(x) = \begin{cases} 0, & x \in [0,1] \\ 1, & x \in [2,3] \end{cases} \end{equation*}

(d) \begin{equation*} f(x) = \begin{cases} \frac{1}{2} + \frac{3}{2} x, & 0 < x \leq \frac{1}{3} \\ 1 - 3(x-\frac{1}{3}), & \frac{1}{3} \leq x \leq \frac{2}{3} \\ \frac{3}{2}(x-\frac{2}{3}), & \frac{2}{3} \leq x < 1 \end{cases} \end{equation*}
4. (15 points) Write a careful proof of the following two theorems:

Theorem: Let \([a, b] \subseteq \mathbb{R}\) be a closed interval and \(f: [a, b] \to \mathbb{R}\) a Riemann integrable function. Then the function \(F: [a, b] \to \mathbb{R}\) defined by

\[ F(x) = \int_a^x f(t) \, dt \]

is continuous.

Theorem: Let \([a, b] \subseteq \mathbb{R}\) be a closed interval and \(f: [a, b] \to \mathbb{R}\) a real-valued function which is continuous on \([a, b]\) and differentiable on \((a, b)\). Prove that if \(f'(x) > 0\) for all \(x \in (a, b)\), then \(f\) is strictly increasing, i.e., if \(a \leq x_1 < x_2 \leq b\), then \(f(x_1) < f(x_2)\).

1. Since \(f\) is integrable, it is bounded, so \(|f(t)| \leq M\) for some \(M > 0\) and all \(t \in [a, b]\). Now if \(x' > x\), then

\[ |F(x') - F(x)| = |\int_x^{x'} f(t) \, dt| \leq \int_x^{x'} |f(t)| \, dt \leq M (x' - x). \]

Hence for all \(x, x' \in [a, b]\) we have

\[ |F(x') - F(x)| \leq M |x' - x|. \]

Given \(\varepsilon > 0\) choose \(\delta = \varepsilon / M\) so that if \(|x' - x| < \delta\) then

\[ |F(x') - F(x)| < \varepsilon. \] This proves that \(F\) is uniformly continuous.

2. By the mean value theorem applied to \(f\) restricted to \([x_1, x_2]\) we have, for some \(t \in (x_1, x_2)\), that

\[ f(x_2) - f(x_1) = f(t) (x_2 - x_1) > 0, \]

from which \(f(x_2) > f(x_1)\).
5. (15 points)

(a) Let $X$ be a metric space and $f_n : X \to \mathbb{R}$ be a sequence of functions. State carefully the definition of "$f_n$ converges uniformly to $f$ as $n \to \infty$", where $f : X \to \mathbb{R}$.

(b) Let $f_n : [0, 1/2) \to \mathbb{R}$ be the function $f_n(x) = \frac{x^n}{2 - x^n}$. Prove that $\{f_n\}$ converges uniformly as $n \to \infty$ and find the limit function $f$.

(c) Now consider $f_n : [0, 1] \to \mathbb{R}$ given by $f_n(x) = \frac{x^n}{2 - x^n}$. Prove that $\{f_n\}$ does not converge uniformly as $n \to \infty$.

(a) For any $\varepsilon > 0$ there exists a positive integer $N$ such that if $n \geq N$ then $|f(x) - f_n(x)| < \varepsilon$ for all $x \in X$.

(b) The limit is the constant function $f(x) = 0$. For the proof, note that for $x \in [0, 1/2]$ we have $x^n \leq \frac{1}{2^n}$. So

$$0 \leq f_n(x) \leq \frac{1/2^n}{2 - 1/2^n} = \frac{1}{2^n - 1} < \frac{1}{2^n}.$$

Given $\varepsilon > 0$ we can find $N \in \mathbb{N}$ so that $\frac{1}{2^n} < \varepsilon$. Then if $n \geq N$ we have $|f_n(x)| < \varepsilon$ for all $x$.

(c) The pointwise limit is

$$f(x) = \begin{cases} 0, & x < 1 \\ 1, & x = 1 \end{cases}$$

which is not continuous. If $\{f_n\}$ converged to $f$ uniformly, then $f$ would be continuous.
6. (10 points) (Extra Credit) Write a careful proof of the following theorem:

**Theorem:** Let \( f_n : X \to \mathbb{R} \) be a sequence of uniformly continuous functions on a metric space \( X \). Suppose \( f_n \to f \) converges uniformly to a function \( f : X \to \mathbb{R} \). Then \( f : X \to \mathbb{R} \) is uniformly continuous.

Given \( \varepsilon > 0 \), choose \( N \) so that \( |f(x) - f_N(x)| < \frac{\varepsilon}{3} \) for all \( x \in X \). Then, since \( f_N \) is uniformly continuous, choose \( \delta \) so that if \( \|x-y\| < \delta \) then \( |f_N(x) - f_N(y)| < \frac{\varepsilon}{3} \) for all \( x, y \in X \).

Then for all \( x, y \in X \),

\[
|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon .
\]

So \( f \) is uniformly continuous.
7. (10 points) (Extra Credit) Estimate the value of the integral
\[ \int_0^1 \frac{x^5}{5 + x^{10}} \, dx \]
with an error of at most .01. Prove an error estimate justifying your answer.

Expand
\[ \frac{x^5}{5 + x^{10}} = \frac{x^5}{5} \cdot \frac{1}{1 + x^{\frac{10}{5}}} \]
\[ = \frac{x^5}{5} \sum_{n=0}^{\infty} (-1)^n \frac{x^{10n}}{5^n} \]
\[ = \sum_{n=0}^{\infty} (-1)^n \frac{x^{10n+5}}{5^{n+1}} = \frac{x^5}{5} - \frac{x^{15}}{5^2} + \frac{x^{25}}{5^3} - \ldots \]

This converges uniformly on \([0, 1]\) so can be integrated term by term:
\[ \int_0^1 \frac{x^5}{5 + x^{10}} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}} \int_0^1 x^{10n+5} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}(10n+6)} \]
\[ = \frac{1}{5 \cdot 6} - \frac{1}{5^2 \cdot 16} + \frac{1}{5^3 \cdot 26} - \ldots \]

This is an alternating series whose terms decrease in absolute value, so the error of a finite truncation is less than the absolute value of the next term. Since \( \frac{1}{5^2 \cdot 16} < .01 \), we can estimate
\[ \int_0^1 \frac{x^5}{5 + x^{10}} \, dx \approx \frac{1}{80} \]
to within .01.
8. (10 points) (Extra Credit) Let \((a, b) \subset \mathbb{R}\) be an open interval and \(f: (a, b) \to \mathbb{R}\) a twice differentiable function with \(f'(x), f''(x) > 0\) for all \(x \in (a, b)\). Then prove that an inverse function \(g\) exists (with domain the image of \(f\)) and \(g\) is twice differentiable. Derive an expression for \(g''\).

Since \(f'(x) > 0\) for all \(x \in (a, b)\), \(f\) is monotonic increasing so invertible. Therefore an inverse \(g\) exists with domain the image of \(f\). By the definition of the inverse

\[
g(f(x)) = x
\]

for all \(x \in (a, b)\). Differentiate using the chain rule:

\[
g'(f(x)) \cdot f'(x) = 1, \quad x \in (a, b).
\]

Differentiate again:

\[
g''(f(x)) \cdot f'(x)^2 + g'(f(x))f''(x) = 0.
\]

From (1), \(g'(f(x)) = \frac{1}{f'(x)}\). Substituting into (2),

\[
g''(f(x)) \cdot f'(x)^2 = \frac{-f''(x)}{f'(x)},
\]

or \(g''(f(x)) = -\frac{f''(x)}{f'(x)^3}\). But this argument assumes \(g', g''\) exist,

which we now show.
For \( g' \neq \frac{d}{dx} g(x) \) and then, since \( f \) is strictly increasing and continuous,

\[
\lim_{h \to 0} \frac{g(f(x)+h) - g(f(x))}{h} = \lim_{h \to 0} \frac{g(f(x+h)) - g(f(x))}{f(x+h) - f(x)} = \frac{f'(x)}{f'(x)}.
\]

For \( g'' \), we similarly have

\[
\lim_{h \to 0} \frac{g'(f(x)+h) - g'(f(x))}{h} = \lim_{h \to 0} \frac{g'(f(x+h)) - g'(f(x))}{f(x+h) - f(x)} = \frac{1}{f'(x)}.
\]

For fixed \( h \), choose by the mean value theorem \( x', x'' \) between \( x \) and \( x+h \) so that

\[
\frac{f(x+h) - f(x)}{h} = f'(x')
\]

and

\[
\frac{f'(x+h) - f'(x)}{h} = f''(x'').
\]

Then

\[
\frac{g'(f(x+h)) - g'(f(x))}{f(x+h) - f(x)} = \frac{1}{f'(x+h)} - \frac{1}{f'(x)} = \frac{-h f''(x'')}{h f'(x') f'(x+h) f'(x)}
\]

Take \( h \to 0 \), and so also \( x', x'' \to x \), to see that the limit exists and given

\[
\frac{-\frac{f''(x)}{f'(x)^3}}.
\]