These “selected solutions” are mostly a few hints and outlines of arguments. You will want to imitate the mathematical writing in the book more than the writing here, which often is a hint rather than a solution.

Problems in Rudin

7. (a) Use the identity \( b^n - 1 = (b - 1)(b^{n-1} + b^{n-2} + \cdots + 1) \), which you can prove by induction, and since \( b > 1 \) we have \( b^j > 1 \) for \( j = 1, 2, \ldots, n - 1 \). Thus the last factor on the right hand side is \( > n \). To attack a problem like this you might start by trying special cases: \( n = 1 \), then \( n = 2 \), then \( n = 3 \). Often extreme cases are illuminating, here small values of \( n \).

(b) Substitute \( b^{1/n} \) for \( b \) in the previous estimate.

(c) If \( t \leq b^{1/n} \), then from (b) and \( t > 1 \) we have

\[
    n \leq \frac{b - 1}{b^{1/n} - 1} \leq \frac{b - 1}{t - 1}
\]

which contradicts the assumption that \( n > (b - 1)/(t - 1) \). Thus \( t > b^{1/n} \).

(d) Straightforward from the hint.

(e) Apply (c) with \( t = b^w \cdot y^{-1} \).

(f) From part (d) if \( b^x < y \), then \( x \) is not an upper bound of \( A \). From part (e) if \( b^x > y \), then \( x \) is not the least upper bound.

(g) If \( b^x = b^{x'} \) and \( x' > x \), then write \( x' = x + \delta \) for some \( \delta > 0 \). Multiplying both sides of \( b^x = b^{x+\delta} \) by \( 1/b^x \) we conclude \( b^\delta = 1 \). Since \( \delta > 0 \) and the rationals are dense in the reals, there is a rational number \( m/n \) with \( 0 < m/n < \delta \). By problem #6, on which this problem relies, we have \( b^\delta > b^{m/n} \), and \( b^{m/n} = (b^m)^{1/n} > 1 \), which is a contradiction.
Other Problems

2. (a) Notice in the preamble we need to allow that \( p \) be the zero polynomial, which is the additive identity in \( F \). The multiplicative identity element is the constant polynomial 1. The addition is

\[
p + p' = \frac{pq + p'q}{qq'}
\]

and the multiplication \((p/q) \cdot (p'/q') = pp'/qq'\). The verification of the field axioms is tedious.

(b) [left purposely blank]

(c) A positive integer \( n \) is viewed as the constant polynomial, so in the equation in (b) is represented by the rational function with \( n = m = 0, \ p_0 = n, \ q_0 = 1 \). Then, for example, the rational function \( f(x) = x \) satisfies \((f - n)(x) = x - n\) and so \( f - n > 0 \) for all \( n \). This violates the Archimedean property.

3. A function \( X \rightarrow Y \) is a subset \( S \subset X \times Y \) such that for \( x \in X \) there exists a unique \( y \in Y \) such that \((x, y) \in S\). Usually we denote a function by a symbol such as \( f \) and write \( f(x) = y \iff (x, y) \in S\). Now if \( S \in X \times Y \) is a function, and \( T \in Y \times Z \) is a function, then we define \( T \circ S \subset X \times Z \) by

\[
T \circ S = \{(x, z) \in X \times Z : \text{there exists } y \in Y \text{ such that } (x, y) \in S \text{ and } (y, z) \in T\}.
\]

We must check that \( T \circ S \) is a function, i.e., that for any \( x \in X \) there is a unique \( z \in Z \) such that \((x, z) \in T \circ S\). Given \( x \) there is a unique \( y \in Y \) so that \((x, y) \in S\) and now for that \( y \) there is a unique \( z \in Z \) such that \((y, z) \in T\). By the definition of \( T \circ S \) we then have \((x, z) \in T \circ S\), and this \( z \) is uniquely determined from \( x \).

4. The domain of \( f \) is \((-1, 1) \subset \mathbb{R}\). The codomain is \( \mathbb{R}\). The range (or image) of \( f \) is \([0, 1) \subset \mathbb{R}\). \( f \) is not injective: \( f(-1/2) = f(1/2) \), for example. \( f \) is not surjective as \( f^{-1}(2) = \emptyset \), for example. \( f \) is neither injective nor surjective, so certainly not bijective. \( f^{-1}([-1/3, 1/2]) = [0, 1/4]. \)

\[
\#
\begin{align*}
f^{-1}(y) &= \left\{ \begin{array}{ll} 0, & y < 0, y \geq 1; \\ 1, & y = 0; \\ 2, & 0 < y < 1. \end{array} \right.
\end{align*}
\]

1 is not in the domain of \( f \), so \( f(1) \) is undefined.

5. I believe the following works and I leave it to you to check carefully: drop me a note if you do or come by and tell me!

Fix a point \( p \in \mathbb{E}^2 \) and suppose \( L_1, L_2 \subset \mathbb{E}^2 \) are two lines. We define \( \delta = d(L_1, L_2) \). Let \( \delta_i \) equal the distance from \( p \) to \( L_i \). (It is zero if \( p \) lies on \( L_i \).) If \( L_1 \) and \( L_2 \) are parallel, and \( p \) lies in
the region between the lines, set \( \delta = \delta_1 + \delta_2 \); if \( p \) lies outside that region, set \( \delta = |\delta_1 - \delta_2| \). If \( L_1, L_2 \) intersect, and if \( \theta \) is the angle between the lines measured in the region containing \( p \), then set \( \delta = \min(\theta + \delta_1 + \delta_2, \pi - \theta + |\delta_1 - \delta_2|) \).

You can think of rotating one of the lines about \( p \) until it is parallel with the other line. There are two possibilities, and those motivate the formula with the minimum. For those who participated in the office discussion, you might interpret these formulas in terms of our discussion.