Test # 2
M365C: Analysis I
November 10, 2016

True/False. _____/30 1. _____/25 2. _____/25 3. _____/20 Total Score. _____/100

Please put your name at the top of the exam. Read over the entire exam before you begin.
Continue your work on the backs of pages or on extra sheets. If your solution runs over onto these
pages, please indicate that clearly. If you use extra sheets, be sure to staple them to the rest of
your exam.

Only attempt the extra credit problems after completing the rest of the exam. Extra credit
problems are scored separately and are not included in the Total Score.

Write neatly! The more neat work you show, the more (partial) credit you will receive.
Have fun!

Answer the following true/false questions. If you find a question ambiguous, write
a brief explanation. Each true/false question is worth 3 points.

True   False

1. Let $X, Y, Z$ be metric spaces. Suppose $f: X \to Y$ and $g: Y \to Z$ are functions
   such that $f$ and $g \circ f$ are continuous. Then $g$ is continuous.

2. The equation $-2x^3 - 3x + 5 = 0$ has a real solution.

3. Let $f: X \to \mathbb{R}$ be a continuous real-valued function defined on a compact metric
   space $X$. Then there exists $p \in X$ such that $f(q) \leq f(p)$ for all $q \in X$.

4. Every monotonically increasing sequence of real numbers converges.

5. Let $A \subset \mathbb{E}^2$ be a connected subset and $p, p' \in A$. Then every point on the line
   segment connecting $p$ and $p'$ is in $A$.

6. Fix $a < b$ and let $f: [a, b] \to \mathbb{E}^2$ be a continuous function which is differentiable
   on $(a, b)$. Then there exists $t \in (a, b)$ such that

$$f'(t) = \frac{f(b) - f(a)}{b - a}.$$

7. The power series $\sum_{n=1}^{\infty} x^n/n^n$ converges for all $x \in \mathbb{R}$. 


1
8. Fix \( a < b \), and let \( f : [a, b] \to \mathbb{R} \) be a continuous function. Then \( f \) is uniformly continuous.

9. Let \( f : (0, 3) \to \mathbb{R} \) be a differentiable function with \( f'(1) = -2 \) and \( f'(2) = 3 \). Then \( f \) has a local minimum in its domain.

10. The function \( f(x) = |x|^2 \), defined on the entire real line, is twice differentiable at \( x = 0 \).
1. (25 points) Consider the following sequences \( \{x_n\}_{n=1}^{\infty} \) of real numbers:

A: \( x_n = (-1)^n \)

B: \( x_n = \sum_{k=0}^{n} \frac{1}{2^k} \)

C: \( \{x_n\} = \left\{ 0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \cdots \right\} \)

D: \( x_n = 3 \frac{n^2 - 3}{n^3 + 1} - \frac{(-1)^n}{\sqrt{n}} \)

E: \( \{x_n\} = \left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \cdots \right\} \)

For each sequence determine if there is a limit as \( n \to \infty \) (Y or N), what the limit is if it exists (leave blank if no limit), the upper limit or lim sup, the lower limit or lim inf, and the set of subsequential limits. Recall that the set of subsequential limits is the set of all possible limits of all convergent subsequences.

<table>
<thead>
<tr>
<th></th>
<th>Limit?</th>
<th>( \lim_{n \to \infty} x_n )</th>
<th>( \limsup_{n \to \infty} x_n )</th>
<th>( \liminf_{n \to \infty} x_n )</th>
<th>Subsequential Limits</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>N</td>
<td></td>
<td>1</td>
<td>-1</td>
<td>{ -1, 1 }</td>
</tr>
<tr>
<td>B</td>
<td>Y</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>{ 2 }</td>
</tr>
<tr>
<td>C</td>
<td>N</td>
<td></td>
<td>1</td>
<td>0</td>
<td>[ 0, 1 ]</td>
</tr>
<tr>
<td>D</td>
<td>Y</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>{ 0 }</td>
</tr>
<tr>
<td>E</td>
<td>N</td>
<td></td>
<td>1</td>
<td>-1</td>
<td>[ -1, 1 ]</td>
</tr>
</tbody>
</table>
2. (25 points)

(a) State carefully the mean value theorem for a function $f: [a, b] \to \mathbb{R}$.

(b) Suppose $f, g: [0, 2] \to \mathbb{R}$ are differentiable at $x = 1$. Compute the derivative of the function $h(x) = f(x^2)g(x)$ at $x = 1$. What is a possible domain for $h$?

(c) Construct a twice differentiable function $f: [0, 3] \to \mathbb{R}$ which satisfies $f(0) = 1$, $f'(1) = -1$, and $f''(2) = 2$.

(a) Suppose $f: [a, b] \to \mathbb{R}$ is continuous and its restriction to $(a, b)$ is differentiable. Then there exists $x \in (a, b)$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

(b) $h'(x) = \frac{d}{dx} \left[ f(x^2)g(x) \right] = f'(x^2)g(x) + f(x^2)g'(x)$

$$h'(1) = 2f'(1)g(1) + f(1)g'(1)$$

The maximal domain is $[0, \sqrt{2}]$.

(c) Since we have 3 conditions, it is natural to try a quadratic function $f(x) = q_0 + q_1x + q_2x^2$. The conditions imply

$q_0 = 1$

$q_1 + 2q_2 = -1$

$2q_2 = 2$

from which

$$f(x) = 1 - 3x + x^2.$$
3. (20 points)

(a) Let \( f: X \to Y \) be a map of metric spaces and \( x \in X \). State carefully the definition of "\( f \) is continuous at \( x \)."

(b) Write a careful proof of the following.

**Theorem:** Let \( X,Y \) be metric spaces, \( f: X \to Y \), and \( x \in X \). Then \( f \) is continuous at \( x \) if and only if for every sequence \( \{x_n\} \) in \( X \) which converges to \( x \) the sequence \( \{f(x_n)\} \) in \( Y \) converges to \( f(x) \).

(a) \( f \) is continuous at \( x \) if for all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( f(B_\delta(x)) \subset B_\varepsilon(f(x)) \).

(b) If \( f \) is continuous at \( x \) and \( x_n \to x \), then given \( \varepsilon > 0 \) choose \( \delta > 0 \) such that \( f(B_\delta(x)) \subset B_\varepsilon(f(x)) \) and \( N \in \mathbb{Z}^+ \) such that \( x_n \in B_\delta(x) \) if \( n \geq N \). It follows that \( f(x_n) \in B_\varepsilon(f(x)) \) if \( n \geq N \). Hence \( f(x_n) \to f(x) \).

Conversely, if \( f \) is not continuous at \( x \) then there exists \( \varepsilon > 0 \) so that for all \( \delta > 0 \) there exists \( x_\delta \in B_\delta(x) \) such that \( f(x_\delta) \notin B_\varepsilon(f(x)) \). Choose such an \( \varepsilon \) and also an \( x_\delta \) for each \( \frac{1}{n} \), \( n \in \mathbb{Z}^+ \). Then we claim (i) \( x_n \to x \) and (ii) \( f(x_n) \not\to f(x) \). Claim (i) follows from the definition of convergence. Given \( \varepsilon > 0 \) choose \( N > \frac{1}{\varepsilon} \) and then \( x_n \in B_\varepsilon(x) \) for all \( n \), for all \( n \geq N \). For (ii) we have \( f(x_n) \notin B_\varepsilon(f(x)) \) for all \( n \), for all but finitely many \( f(x_n) \) are contained in any ball about \( f(x) \).
4. (10 points) (Extra Credit) Write a careful proof of the following.

**Theorem:** Let $X, Y$ be metric spaces and $f : X \to Y$ a continuous function. Suppose $X$ is compact. Then $f$ is uniformly continuous.

If $f$ is not uniformly continuous, then there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exists $x \in X$ such that $f(B_\delta(x)) \not\subset B_\varepsilon(f(x))$.

Setting $\delta = \frac{1}{n}$, we fix a sequence $\{x_n\} \subset X$ such that $B_{\frac{1}{n}}(x_n) \not\subset B_\varepsilon(f(x_n))$. Since $X$ is compact there is a convergent subsequence $x_{n_i} \to x_0$. Since $f$ is continuous, there exists $\delta > 0$ such that $f(B_\delta(x_0)) \subset B_{\varepsilon/2}(f(x_0))$.

Choose $\varepsilon > 0$ so that $d(f(x_{n_i}), f(x_0)) < \varepsilon/2$.

If $x' \notin B_{\frac{1}{n_i}}(x_{n_i})$, then

$$d(x_0, x') \leq d(x_0, x_{n_i}) + d(x_{n_i}, x') < \frac{\varepsilon}{2} + \frac{1}{n_i} < \delta.$$ 

Thus we have $d(f(x_0), f(x')) < \varepsilon/2$, by (i).

But

$$d(f(x_{n_i}), f(x')) \leq d(f(x_{n_i}), f(x_0)) + d(f(x_0), f(x')) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$ 

This contradicts $B_{\frac{1}{n_i}}(x_{n_i}) \not\subset B_\varepsilon(f(x_{n_i}))$. 


5. (10 points) (Extra Credit) Suppose \( \{a_n\} \) is a sequence of positive real numbers such that \( \sum a_n \) diverges. Prove that \( \sum \frac{a_n}{1+a_n} \) also diverges. What can you say about \( \sum \frac{a_n}{1+n^2} \)?

If \( a_n \to 0 \) then \( \frac{a_n}{1+a_n} \to 0 \) and so \( \sum \frac{a_n}{1+a_n} \) diverges.

If \( a_n \to 0 \) then for some \( N > 0 \) we have \( a_n < \frac{1}{2} \) if \( n > N \).

Then for \( n > N \) also \( \frac{a_n}{1+a_n} > \frac{a_n}{1+\frac{1}{2}} = \frac{2}{3} a_n \). Thus

\[
\sum_{n=N}^{M} \frac{a_n}{1+a_n} > \frac{2}{3} \sum_{n=N}^{M} a_n
\]

and since \( M \to \infty \) the RHS diverges. So the LHS diverges.

For \( a_n = \frac{1}{n} \), \( \frac{a_n}{1+n^2} = \frac{1}{2n} \), so \( \sum \frac{a_n}{1+n^2} \) diverges.

For \( a_n = \frac{1}{n} \), \( \frac{1}{2} \), \( \frac{1}{3} \), \( \frac{1}{8} \), \( \frac{1}{12} \), \( \frac{1}{16} \), \( \frac{1}{2^4} \), \( \frac{1}{8} \), \( \frac{1}{12} \), \( \frac{1}{16} \), \( \cdots \)

we have

\[
\left(1 - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{12}\right) + \left(\frac{1}{7} - \frac{1}{16}\right) + \cdots
\]

\[
> \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{1}{12}\right) + \left(\frac{1}{8} - \frac{1}{16}\right) + \cdots
\]

\[
= \frac{1}{4} + \frac{1}{8} + \frac{1}{12} + \frac{1}{16} + \cdots
\]

\[
= \frac{1}{4} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right)
\]

diverges. But \( \sum \frac{a_n}{1+n^2} = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots \) converges.
6. (10 points) (Extra Credit) Write a careful proof of the following.

**Theorem:** Let \( f: [a, b] \to \mathbb{R} \) be a continuously differentiable function. Prove that \( f \) is uniformly differentiable in the sense that for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( t, x \in [a, b] \) with \( 0 < |t - x| < \delta \), then

\[
\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \varepsilon.
\]

The hypotheses mean that \( f \) is defined and differentiable on an open interval containing \([a, b]\), and the derivative \( f' \) is continuous on \([a, b]\).

Since \( f' \) is continuous on a compact space, it is uniformly continuous. Hence there exists \( \delta > 0 \) so that \( |y - x| < \delta \) implies \( |f'(y) - f'(x)| < \varepsilon \), where \( y, x \in [a, b] \). If now \( t, x \in [a, b] \) with \( 0 < |t - x| < \delta \), then by the MVT there exists \( y \) between \( t \) and \( x \) such that

\[
f'(y) = \frac{f(t) - f(x)}{t - x}.
\]

Since \( |y - x| < |t - x| < \delta \), we have \( |f'(y) - f'(x)| < \varepsilon \). Substitute (1).