Problem Set # 13

M382E: Algebraic Topology

Due: December 4, 2008

This homework is due December 4, the last day of class. For further study on characteristic classes I recommend the book by Milnor and Stasheff as well as the last chapter of Bott and Tu. You will enjoy reading all of Bott and Tu.

Problems

1. (a) Let $E \to X$ be an $\mathbb{F}$-vector bundle of rank n over a space X, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. Construct a principal $GL_n\mathbb{F}$-bundle $\mathcal{F}(E) \to X$, the *bundle of frames*, and express the vector bundle E as a bundle associated to some linear action of $GL_n\mathbb{F}$ on a vector space. A point of $\mathcal{F}(E)_x$ $(x \in X)$ has a geometric meaning: it is a linear isomorphism $\mathbb{F}^n \to E_x$.

   (b) Express the dual $E^* \to X$ as a bundle associated to $\mathcal{F}(E) \to X$. What about $E \otimes E$? $\bigwedge^\bullet E$?

   (c) If $E' \to X$ and $E'' \to X$ are vector bundles, construct the direct sum $E' \oplus E'' \to X$ and the tensor product $E' \otimes E'' \to X$.

2. Let $E' \to X$ and $E'' \to X$ be oriented real vector bundles. Let $E = E' \oplus E''$ have the direct sum orientation: if for some $x \in X$ we have $\{e'_1, \dots, e'_{n'}\}$ is an oriented basis of E'_x and $\{e''_1, \dots, e''_{n''}\}$ is an oriented basis of E''_x, then $\{e'_1, \dots, e'_{n'}, e''_1, \dots e''_{n''}\}$ is an oriented basis of $(E'_x \oplus E''_x) = E'_x \oplus E''_x$. Derive a formula for $e(E' \oplus E'')$ in terms of $e(E')$ and $e(E'')$. Be careful with signs!

3. Let M be a smooth oriented manifold and $N \subset M$ a smooth compact oriented submanifold. Recall that there exists a *tubular neighborhood*: an open set $E \subset M$ containing N and a vector bundle $\pi\colon E \to N$ with an isomorphism to the normal bundle of N in M. How is the Thom class of E related to the Poincaré dual of $[N]$?

4. (a) Recall the Hopf bundle $S(V) \to \mathbb{P}(V)$, where V is a complex vector space with Hermitian inner product, $S(V) \subset V$ the sphere of unit vectors, and $\mathbb{P}(V)$ the projective space. This is a principal $\mathbb{T}$-bundle, where $\mathbb{T} \subset \mathbb{C}$ is the group of unit norm complex numbers (under multiplication). Compute the Euler class of the complex line bundle associated to the action of $\mathbb{T}$ on $\mathbb{C}$ by scalar multiplication.

   (b) Now let W be a real vector space with inner product and $S(W) \to \mathbb{P}(W)$ the associated Hopf bundle. It is a principal $\mathbb{Z}/2\mathbb{Z}$-bundle, where $\mathbb{Z}/2\mathbb{Z} = \{\pm 1\} \subset \mathbb{R}$. Consider the real 2-plane bundle associated to the diagonal action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{R} \oplus \mathbb{R}$ by scalar multiplication. Show that it is oriented and compute its Euler class.

   (c) How are the bundles in parts (a) and (b) related? (You can tackle this problem without solving (a) and (b).)