Problem Set # 2

M382E: Algebraic Topology

Due: September 16, 2008

After the last lecture on bordism, we begin to follow more closely Hatcher, Chapter 2. Please begin to read it! The motivational discussion at the beginning of the chapter is useful, and I won't repeat it in lecture. You may also want to read Chapter 0 over the next few weeks as there is much useful material there. I also posted an elementary introduction to simplicial sets (which we won't study) which includes in §2.3 a definition of Δ-sets and some of the ideas discussed in lecture. Hatcher's treatment does not bring out the combinatorial nature as explicitly as we do and as is done in this paper.

For the problems in Hatcher on Δ-complexes, present the indicated Δ-complex as the geometric realization of a Δ-set.

Problems in Hatcher

Section 2.1 (page 131): 2, 3, 5, 9, 14, 30

Other Problems

1. Recall that the cone CX of a topological space X is the quotient $CX = [0,1]\times X \;/\; \{0\}\times X$. Define the *suspension* SX as the quotient $SX = CX \;/\; \{1\}\times X$. Define a natural inclusion $X \hookrightarrow SX$ with image $\{1/2\}\times X$.
   (a) Draw a schematic picture of the suspension. What is the suspension of S^0? (That is, can you recognize it as homeomorphic to a familiar space? Prove it is so.) Of S^1? Of S^n?
   (b) Use Mayer-Vietoris to relate $\widetilde{MSO}_\bullet(X)$ and $\widetilde{MSO}_\bullet(SX)$.

2. (a) Suppose h is MO or MSO. Let (X, A) be a pair of topological spaces, i.e., X is a topological space and $A \subset X$ a subspace. For convenience, assume $A \neq \emptyset$. By definition $h_\bullet(X,A) = \tilde{h}_\bullet(X,A)$ is defined to be $\tilde{h}_\bullet(X \cup CA)$. Use Mayer-Vietoris to derive the long exact sequences

   $$\cdots \longrightarrow h_n(A) \longrightarrow h_n(X) \longrightarrow h_n(X,A) \longrightarrow h_{n-1}(A) \longrightarrow \cdots$$

   and

   $$\cdots \longrightarrow \tilde{h}_n(A) \longrightarrow \tilde{h}_n(X) \longrightarrow \tilde{h}_n(X,A) \longrightarrow \tilde{h}_{n-1}(A) \longrightarrow \cdots$$

   Describe the various maps in the sequence.
   (b) We say A is a *retract* of X if there is a map $f\colon X \to X$ (a *retraction*) with $f(X) = A$ and $f\big|_A = \mathrm{id}_A$. What can you say about the exact sequence in (a) in case A is a retract of X?
   (c) Show that a deformation retraction gives rise to a retraction. Give an example of a retraction which is not a deformation retraction.

3. (a) Use the classification of 1- and 2-manifolds to prove that $MSO_q(\text{point}) = 0$ for $q = 1, 2$. Prove that $\widetilde{MSO}_{n+q}(S^n) = 0$ for all n and $q = 1, 2$.

   (b) Recall that $\mathbb{CP}^n$ is the smooth $2n$-dimensional manifold defined as the quotient of $\mathbb{C}^{n+1} = \{(z^0, z^1, \dots, z^n) : z^i \in \mathbb{C}\} \setminus \{(0, 0, \dots, 0)\}$ by the equivalence relation

$$(z^0, z^1, \dots, z^n) \sim (\lambda z^0, \lambda z^1, \dots, \lambda z^n), \qquad \lambda \in \mathbb{C} \setminus \{0\}.$$

   Show that we get the same space by restricting to $|z^0|^2 + |z^1|^2 + \cdots + |z^n|^2 = 1$ and $|\lambda| = 1$.

   (c) Use induction and Mayer-Vietoris to compute $\widetilde{MSO}_q(\mathbb{CP}^2)$ for $q \leq 4$.

4. Compute the homology of the Δ-set described in Example 2.5 on page 107 of Hatcher, at least for the cases $n = 1, 2, 3$.

5. (a) Let $\mathcal{C}$ be a small category (see Problem Set #1). Form a Δ-set $N\mathcal{C}$, called the *nerve* of $\mathcal{C}$, by letting $N\mathcal{C}_n$ be the set of compositions of n-morphisms in $\mathcal{C}$. Thus an element of $N\mathcal{C}_n$ may be denoted $(a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} a_n)$, where a_i are objects and f_i are morphisms. $N\mathcal{C}_0$ is the set of objects. Carefully define the boundary maps $d_0, d_1, \dots, d_n$ using composition: d_i omits a_i. For example,

$$d_0(a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} a_2) = (a_1 \xrightarrow{f_2} a_2)$$
$$d_1(a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} a_2) = (a_0 \xrightarrow{f_2 \circ f_1} a_2)$$
$$d_2(a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} a_2) = (a_0 \xrightarrow{f_1} a_1)$$

   Check the relations in the definition of a Δ-set.

   (b) Apply this construction to the category with two objects a, b and three morphisms: in addition to the identity morphisms there is a unique morphism $a \to b$. Consider the chain complex attached to the resulting Δ-set. Can you compute its homology? At least in low degrees? What can you say about the geometric realization of this Δ-set?

   (c) Now construct a category $\mathcal{C}$ attached to a group G: it has a single object $*$ and $\mathcal{C}(*, *) = G$ with the associative composition from G. (We don't use all the structure in G in this construction, so we can generalize to a category attached to a *monoid*.) Consider the simplest nontrivial case: G cyclic of order two. Again make the Δ-set and corresponding chain complex. Homology? Low degrees?

   (d) Consider the chain complex C associated to the Δ-set in part (a). Let D be the subcomplex generated by elements $(a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} a_n)$ where at least one of the f_i is the identity map. Show that it *is* a subcomplex, i.e., is closed under the boundary operator. Therefore construct a quotient cochain complex $C' = C/D$. Write down C' for the examples in parts (b) and (c) and compute its homology.