

Problem Set # 7

M382E: Algebraic Topology

Due: October 21, 2008

You'll need to read the material on Euler characteristics in Hatcher, pages 146–7.

Problems in Hatcher

Section 2.2 (page 155): 20, 22, 23, 40, 41

Other Problems

- Let $\pi: Y \rightarrow X$ be a Galois (regular, normal, principal) covering space with group Γ of deck transformations.
 - Suppose X has a CW structure. Use the pullback under π to construct a CW structure on Y such that Γ acts by permuting the cells.
 - Let $\mathbb{Z}\Gamma$ denote the *group ring* of Γ . It is a ring (with unit) whose elements are formal sums and differences of elements in Γ . The addition is formal, but the multiplication uses the group structure of Γ . Work this out and verify that you get a ring. For $\Gamma \cong \mathbb{Z}$ identify this ring with the ring $\mathbb{Z}[t, t^{-1}]$ of Laurent polynomials.
 - Show that the cellular chain complex of Y constructed in (a) consists of $\mathbb{Z}\Gamma$ -modules and the differential is a module map.
 - Express the cellular chain complex of X in terms of the cellular chain complex of Y . You should find that it is a quotient. Which quotient?
 - Work this out explicitly for the covering $\mathbb{R} \rightarrow S^1$.
 - The Alexander polynomial of a knot is defined using the ideas in this problem. Google if you are interested.
- In this problem you are asked to find explicit cycles which represent homology classes. I want to be sure that you are thinking about homology geometrically at the same time you learn the algebraic tools necessary to compute. Let X be a space. Let A be an abelian group.
 - A homology class in $H_1(X; A)$ may be represented as follows. Let $\gamma_i: S^1 \rightarrow X$ be a finite set of continuous maps and $a_i \in A$. Then any class can be realized as $\sum_i a_i \cdot (\gamma_i)_*(e)$, where e is the standard generator of $H_1(S^1)$. Represent the generator of $H_1(\mathbb{R}P^n)$ so. Represent the elements of $H_1(S^1 \times S^1)$ this way. What is the minimal number of maps γ_i needed? If $A = \mathbb{Z}$ can you make a statement in general about the minimal number of maps?

- (b) Some classes in $H_2(X; A)$ may be represented as the image of the generator of $H_2(S^2)$ under a map $S^2 \rightarrow X$, or a linear combination of multiples thereof. Is this true of classes in $H_2(\mathbb{C}\mathbb{P}^n)$? What about $H_2(\mathbb{R}\mathbb{P}^n; \mathbb{Z}/2\mathbb{Z})$? What about $H_2(S^1 \times S^1)$? $H_2(S^1 \times S^1; \mathbb{Z}/2\mathbb{Z})$?
- (c) Quite generally, a class in $H_q(X)$ is *spherical* if it is the image of the generator of $H_q(S^q)$ under a continuous map $S^q \rightarrow X$. Which classes in $H_\bullet(\mathbb{R}\mathbb{P}^n)$ are spherical?

3. (a) Compute $H^\bullet(\mathbb{C}\mathbb{P}^\infty; A)$ for all abelian groups A .

(b) Compute $H_\bullet(\mathbb{R}\mathbb{P}^n; \mathbb{Z}/n\mathbb{Z})$ for all integers $n > 0$.

(c) Compute $H^\bullet(\mathbb{R}\mathbb{P}^n; \mathbb{Z}/n\mathbb{Z})$ for all integers $n > 0$.