

CUP PRODUCTS AND INTERSECTION NUMBER

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In lecture I sketched the proof of a theorem relating cup product and intersection number via Poincaré duality. This note includes the details and more remarks.

We begin with a local model for intersections. Let $p + q = n$ be nonnegative integers and \mathbb{A}^n standard affine space with

$$(1) \quad \begin{aligned} \mathbb{A}^p &= \{(x^1, \dots, x^p, 0, \dots, 0)\} \subset \mathbb{A}^n \\ \mathbb{A}^q &= \{(0, \dots, 0, x^{p+1}, \dots, x^n)\} \subset \mathbb{A}^n \end{aligned}$$

These subspaces intersect transversely at the origin. We use the standard orientations on $\mathbb{A}^p, \mathbb{A}^q$, and \mathbb{A}^n so that the local intersection number (say, as defined in differential topology) of \mathbb{A}^p and \mathbb{A}^q at the origin is $+1$. Let $B^p \subset \mathbb{A}^p$ and $B^q \subset \mathbb{A}^q$ be the closed balls of radius $1/2$; then $B^p \times B^q \subset \mathbb{A}^n$ is compact and homeomorphic to the n -ball $B^n \subset \mathbb{A}^n$.

Throughout we work with oriented manifolds and \mathbb{Z} coefficients. There is an analogous discussion with unoriented manifolds and $\mathbb{Z}/2\mathbb{Z}$ coefficients.

Let M^n, P^p, Q^q be compact oriented topological manifolds, where $p + q = n$. Suppose $f: P \rightarrow M$ and $g: Q \rightarrow M$ are continuous; then $f_*[P] \in H_p(M)$ and $g_*[Q] \in H_q(M)$ are homology classes represented by these manifolds. Let $\alpha_P \in H^q(M)$ and $\alpha_Q \in H^p(M)$ be the Poincaré duals guaranteed by the Poincaré duality theorem, so that

$$(2) \quad \begin{aligned} f_*[P] &= \alpha_P \frown [M] \\ g_*[Q] &= \alpha_Q \frown [M] \end{aligned}$$

Definition 3. *We say f and g intersect transversely if $f(P) \cap g(Q) = \{x_1, \dots, x_N\} \subset M$ is finite, and about each x_i there is an open neighborhood U_i and a homeomorphism $\varphi_i: U_i \rightarrow \mathbb{A}^n$ such that $\varphi_i(U_i \cap f(P)) = \mathbb{A}^p$; $\varphi_i(U_i \cap g(Q)) = \mathbb{A}^q$; $f^{-1}(U_i)$ and $g^{-1}(U_i)$ are connected; and the maps $\varphi_i \circ f|_{f^{-1}(U_i)}$ and $\varphi_i \circ g|_{g^{-1}(U_i)}$ are orientation-preserving homeomorphisms onto their images.*

Remark 4. The word ‘transverse’ is normally used in the smooth category; I have co-opted it here for an analogous concept in the category of topological manifolds.

For any $i_C: C \hookrightarrow M$ let $(i_C)_*: H_\bullet(M) \rightarrow H_\bullet(M|C)$ denote the restriction, where recall the notation $H_\bullet(M|C) = H_\bullet(M, M - C)$ for the homology localized at C . Define $C_i = \varphi_i^{-1}(B^p \times B^q) \subset U_i$ and define $\epsilon_i = \pm 1$ by the equation

$$(5) \quad (\varphi_i)_*(\iota_{C_i})_*[M] = \epsilon_i \mu,$$

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where

$$(6) \quad \mu \in H_n(\mathbb{A}^n | B^p \times B^q)$$

is given by the standard orientation of \mathbb{A}^n and the fundamental lemma which glues these together into consistent orientations localized on compact sets. The number ϵ_i is the *local intersection number* of f and g at x_i .

The theorem we prove in this note is the following.¹

$$\mathbf{Theorem\ 7.} \quad \langle \alpha_Q \smile \alpha_P, [M] \rangle = \sum_{i=1}^N \epsilon_i.$$

The right hand side is the *global* intersection number, obtained by summing the local intersection numbers. The left hand side is the intersection product of α_Q and α_P ; the theorem justifies the nomenclature ‘intersection product’. Of course, this term is used for any cohomology classes; here we have cohomology classes Poincaré dual to the images of manifolds. We have not assumed that these are submanifolds, but only that there is a finite number of intersection points and near these points the images of these manifolds look like submanifolds with a transverse intersection.

As a preliminary we give standard generators for certain homology groups in our local model $\mathbb{A}^n = \mathbb{A}^p \times \mathbb{A}^q$. Let $\Delta^p \subset \mathbb{A}^n$ be the p -simplex with vertices

$$(8) \quad \begin{aligned} 0 &= (-1, -1, -1, \dots, -1; -1, \dots, -1) \\ 1 &= (+1, -1, -1, \dots, -1; -1, \dots, -1) \\ 2 &= (-1, +1, -1, \dots, -1; -1, \dots, -1) \\ &\dots \\ p &= (-1, -1, -1, \dots, +1; -1, \dots, -1) \end{aligned}$$

and $\Delta^q \subset \mathbb{A}^n$ the q -simplex with vertices

$$(9) \quad \begin{aligned} 0 &= (-1, \dots, -1; -1, -1, -1, \dots, -1,) \\ p+1 &= (-1, \dots, -1; +1, -1, -1, \dots, -1) \\ p+2 &= (-1, \dots, -1; -1, +1, -1, \dots, -1) \\ &\dots \\ p+q = n &= (-1, \dots, -1; -1, -1, -1, \dots, +1) \end{aligned}$$

The semicolon separates the first p coordinates from the last q coordinates. Let $\Delta^{p+q} \subset \mathbb{A}^n$ be the simplex with vertices $\{0, \dots, n\}$.

Lemma 10. *The cohomology groups $H_p(\mathbb{A}^n | B^p \times \mathbb{A}^q)$, $H_q(\mathbb{A}^n | \mathbb{A}^p \times B^q)$, and $H_n(\mathbb{A}^n | B^p \times B^q)$ are infinite cyclic with generators Δ^p , Δ^q , and Δ^{p+q} . The cohomology groups in other degrees vanish.*

¹Our sign conventions for cup and cap products are used, of course. Recall that $\langle \alpha \smile \alpha', \sigma \rangle = (-1)^{pq} \langle \alpha, \rho \sigma \rangle \langle \alpha', \sigma_q \rangle$ if α has degree p and α' has degree q .

Proof. For the last, the pair $(\mathbb{A}^n|B^p \times B^q)$ is homeomorphic to $(\mathbb{A}^n|B^n) = (\mathbb{A}^n, \mathbb{A}^n - B^n)$, which by excision has the same homology as $(\Delta^{p+q}, \partial\Delta^{p+q})$. Earlier in the course we showed that this homology is only nonzero in degree n with generator Δ^{p+q} . For the first two, we use homotopy invariance to collapse the extra affine space and then use excision again.

The key idea in the proof of Theorem 7 is to construct Poincaré dual classes β_P, β_Q which are localized in a neighborhood of the images $f(P), g(Q)$. It may be helpful to keep in mind a de Rham analog in our local model $\mathbb{A}^n = \mathbb{A}^p \times \mathbb{A}^q$. So if x^1, \dots, x^{p+q} are standard local coordinates, then the local Poincaré dual of $\mathbb{A}^p \subset \mathbb{A}^n$ is $\rho(x^{p+1}, \dots, x^{p+q}) dx^{p+1} \wedge \dots \wedge dx^{p+q}$, where $\rho: \mathbb{A}^q \rightarrow \mathbb{R}$ is a function with compact supports such that $\int_{\mathbb{A}^q} \rho(x^{p+1}, \dots, x^{p+q}) dx^{p+1} \wedge \dots \wedge dx^{p+q} = +1$. We can take the support of ρ as small as we like. In particular, this local model for the Poincaré dual may be taken to be supported in $\mathbb{A}^p \times B^q$. In the proof which follows we first construct global compact sets $K, L \subset M$ whose interiors will contain the supports of β_P, β_Q ; the latter are constructed by applying Poincaré duality to those interiors, which are noncompact manifolds. Then the intersection product on the left hand side of Theorem 7 reduces to a sum of intersection products in the local model, and we use the basic adjunction between cup and cap products to evaluate it.

One more preliminary, which you can read about in Hatcher (as recommended at the start of Homework #12), is the relative cap product. If X is a space with open sets $A, B \subset X$, and $p+q = n$, then the relative cap product is a map

$$(11) \quad \frown: H^p(X, A) \times H_n(X, A \cup B) \longrightarrow H^q(X, B).$$

Thus for closed sets $K, L \subset X$ we have a relative cap product

$$(12) \quad \frown: H^p(X|K) \times H_n(X|K \cap L) \longrightarrow H^q(X|L).$$

Proof of Theorem 7. First, we construct² compact sets $K, L \subset M$ such that $\text{int}(K) \supset f(P)$; $\text{int}(L) \supset g(Q)$; $K \cap L \subset \bigcup_i U_i$; $\varphi_i(U_i \cap K) = \mathbb{A}^p \times B^q$; and $\varphi_i(U_i \cap L) = B^p \times \mathbb{A}^q$. Let C be the closed set which is the complement of $\bigcup_i U_i$ in M . Let A, B be open sets which separate the disjoint closed sets $f(P) \cap C, g(Q) \cap C$. This means the respective closed sets are contained in A, B respectively, and $A \cap B = \emptyset$. These sets exist since M is a normal topological space (as we assume it is Hausdorff and paracompact, hence metrizable). Then define

$$(13) \quad \begin{aligned} K &= \overline{\bigcup_i \varphi_i^{-1}(\mathbb{A}^p \times B^q) \cup A} \\ L &= \overline{\bigcup_i \varphi_i^{-1}(B^p \times \mathbb{A}^q) \cup B} \end{aligned}$$

Next, using the Poincaré duality isomorphism $D_{\text{int}(K)}: H_c^q(\text{int}(K)) \rightarrow H_p(\text{int}(K))$ we find $\beta_P \in H_c^q(\text{int}(K))$ such that $D_{\text{int}(K)}(\beta_P) = f_*[P]$. Since the support of β_P is contained in K , we can by

²In a smooth setting the analog of the interiors of K, L are tubular neighborhoods of submanifolds.

excision regard $\beta_P \in H^q(M|K)$. Let $\mu_K = (i_K)_*[M] \in H_n(M|K)$. Then by the naturality of the cap product we deduce

$$(14) \quad f_*[P] = \beta_P \frown \mu_K.$$

Similarly, we construct $\beta_Q \in H^p(M|L)$ with

$$(15) \quad g_*[Q] = \beta_Q \frown \mu_K.$$

Again by naturality we deduce

$$(16) \quad \begin{aligned} \alpha_P &= i_K^* \beta_P \\ \alpha_Q &= i_L^* \beta_Q \end{aligned}$$

where $i_K^*: H^q(M|K) \rightarrow H^q(M)$ and $i_L^*: H^p(M|L) \rightarrow H^p(M)$. Therefore, $\beta_Q \smile \beta_P \in H^n(M|K \cap L)$ and from (16) we have $i_{K \cap L}^*(\beta_Q \smile \beta_P) = \alpha_Q \smile \alpha_P$. Let $j_i: U_i \hookrightarrow M$ denote the inclusion and set

$$(17) \quad \gamma_P^{(i)} = j_i^* \beta_P \in H^q(U_i|U_i \cap K) \cong H^q(\mathbb{A}^n|\mathbb{A}^p \times B^q).$$

Similarly we define $\gamma_Q^{(i)} = j_i^* \beta_Q \in H^p(\mathbb{A}^n|B^p \times \mathbb{A}^q)$. Hence we compute

$$(18) \quad \begin{aligned} \langle \alpha_Q \smile \alpha_P, [M] \rangle &= \langle \beta_Q \smile \beta_P, (i_{K \cap L})_*[M] \rangle \\ &= \sum_{i=1}^N \langle j_i^* \beta_Q \smile j_i^* \beta_P, (i_{U_i \cap K \cap L})_*[M] \rangle \\ &= \sum_{i=1}^N \epsilon_i \langle \gamma_Q^{(i)} \smile \gamma_P^{(i)}, \mu \rangle \end{aligned}$$

In the last line we use the homeomorphisms φ_i without indicating them in the notation. Equation (18) is the promised reduction of the global intersection number to a sum of local intersection numbers, which we now proceed to evaluate.

We use Lemma 10 and so the explicit cycle Δ^{p+q} to represent μ in (6). Note in the notation of that lemma that the front p -face of Δ^{p+q} is Δ^p and the back q -face is Δ^q . Recall the formula for the cup product, which involves a sign and evaluation on the back face. Thus, using the symbol $\gamma_P^{(i)}$ to denote both a cohomology class and a cocycle representative, we have

$$(19) \quad \gamma_P^{(i)} \frown \mu = \gamma_P^{(i)} \frown \Delta^{p+q} = (-1)^{pq} \langle \gamma_P^{(i)}, \Delta^q \rangle \Delta^p.$$

To evaluate the next cap product we let Δ^{q+p} be the simplex with vertices in order $\{p+1, p+2, \dots, p+q, 0, 1, \dots, p\}$; see (8) and (9). The front q -face is $(-1)^q \Delta^q$ and the back p -face is Δ^p . We also have that Δ^{q+p} represents $(-1)^{(p+1)q} \mu$ in homology. Thus

$$(20) \quad \gamma_Q^{(i)} \frown \mu = (-1)^{(p+1)q} \gamma_Q^{(i)} \frown \Delta^{q+p} = (-1)^{(p+1)q} (-1)^{pq} \langle \gamma_Q^{(i)}, \Delta^p \rangle (-1)^q \Delta^q = \langle \gamma_Q^{(i)}, \Delta^p \rangle \Delta^q.$$

Now, by naturality we compute

$$(21) \quad \epsilon_i \gamma_P^{(i)} \frown \mu = j_i^* \beta_P \frown (i_{u_i \cap K \cap L})_* \mu_K = (i_{U_i \cap L})_*(\beta_P \frown \mu_K) = (i_{U_i \cap L})_* f[P] = \Delta^p.$$

We use the cap product (12) in the second expression. The third equality is (14). In the last equality we identify $H_p(M|U_i \cap L) \cong H_p(U_i|U_i \cap L) \cong H_p(\mathbb{A}^n|B^p \times \mathbb{A}^q)$ using excision and φ_i and we use the fact that $\varphi_i \circ f|_{f^{-1}(U_i)}$ is orientation-preserving. Comparing (19) and (21) we find

$$(22) \quad \langle \gamma_P^{(i)}, \Delta^q \rangle = (-1)^{pq} \epsilon_i.$$

Analogously to (21) we compute

$$(23) \quad \epsilon_i \gamma_Q^{(i)} \frown \mu = j_i^* \beta_Q \frown (i_{u_i \cap K \cap L})_* \mu_K = (i_{U_i \cap L})_*(\beta_Q \frown \mu_K) = (i_{U_i \cap L})_* g[Q] = \Delta^q.$$

Thus comparing (20) and (23) we find

$$(24) \quad \langle \gamma_Q^{(i)}, \Delta^p \rangle = \epsilon_i.$$

Finally, we use (18), (22), and (24) and the definition of the cup product to evaluate

$$(25) \quad \begin{aligned} \langle \alpha_Q \smile \alpha_P, [M] \rangle &= \sum_{i=1}^N \epsilon_i \langle \gamma_Q^{(i)} \smile \gamma_P^{(i)}, \mu \rangle \\ &= \sum_{i=1}^N \epsilon_i \langle \gamma_Q^{(i)} \smile \gamma_P^{(i)}, \Delta^{p+q} \rangle \\ &= \sum_{i=1}^N \epsilon_i (-1)^{pq} \langle \gamma_Q^{(i)}, \Delta^p \rangle \langle \gamma_P^{(i)}, \Delta^q \rangle \\ &= \sum_{i=1}^N \epsilon_i (-1)^{pq} \epsilon_i (-1)^{pq} \epsilon_i \\ &= \sum_{i=1}^N \epsilon_i. \end{aligned}$$