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SOME CONSEQUENCES OF A THEOREM OF BOTT

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It will be shown that the following theorem, due to R. Bott [3], can be used to solve several well known problems ; including the problem of the existence of division algebras, and the parallelizability of spheres[2]. (Independent solutions of these problems, also based on Bott's work, have been given by Kervaire and Hirzebruch.)

THEOREM OF BOTT. *For any O_m-bundle ξ over the sphere S^{4k}, the Pontrjagin class $p_k(\xi) \in H^{4k}(S^{4k}; Z)$ is divisible by* $(2k-1)!$. (This result was conjectured, and proved up to powers of 2, by Borel and Hirzebruch [1]).

The following result, which follows from Wu Wen-Tsün [19], will also be needed. Since Wu's paper is in Chinese a proof is included in the appendix. The epimorphism $Z \to Z_4$ induces a homomorphism $H^*(K; Z) \to H^*(K; Z_4)$ which will be denoted by $\alpha \to (\alpha)_4$. Let $i: Z_2 \to Z_4$ denote the inclusion homomorphism.

THEOREM OF WU. *For any O_m-bundle ξ over a complex K, the class $(p_k(\xi))_4 \in H^{4k}(K; Z_4)$ is determined by the Stiefel-Whitney classes $w_l(\xi) \in H^l(K; Z_2)$. In particular if the Stiefel-Whitney classes $w_1(\xi), \cdots, w_{4k-1}(\xi)$ are zero then $(p_k(\xi))_4 = i_* w_{4k}(\xi)$.*

Combining these two results, the following is obtained.

THEOREM 1. *There exists an O_m-bundle ξ over the sphere S^n with $w_n(\xi) \neq 0$ only for $n = 1, 2, 4$ or 8.*

(Examples of such bundles can be given as follows : for $n=1$ the 2-fold covering of the circle, and for $n = 2$, 4 or 8 the O_n-bundle over S^n associated with the Hopf fibering $S^{2n-1} \to S^n$.)

PROOF. According to Wu [16] such a bundle can exists only if n is a power of 2. Hence it is certainly sufficient to consider the case $n = 4k$, $k > 2$. The identity

$$(p_k(\xi))_4 = i_* w_{4k}(\xi) \in H^{4k}(S^{4k}; Z_4) = Z_4$$

is valid, since the lower Stiefel-Whitney classes must be zero. In other words the class $w_{4k}(\xi)$ is zero if and only if $p_k(\xi)$ is divisible by 4. But

[1] The author holds a Sloan fellowship.

[2] A preliminary account of this work has been given in [20].

$p_k(\xi)$ is known to be divisible by $(2k-1)!$. For $k > 2$ this proves that $w_{4k}(\xi) = 0$.

THEOREM 2. *The sphere S^r is parallelizable only for $r = 1, 3, 7$.* (Compare Steenrod and Whitehead [10].)

PROOF. The fibering $SO_r \longrightarrow SO_{r+1} \xrightarrow{f} S^r$ associated with the tangent bundle of S^r has the following homotopy sequence:

$$\longrightarrow \pi_r(SO_{r+1}) \xrightarrow{f_*} \pi_r(S^r) \xrightarrow{\partial} \pi_{r-1}(SO_r) \longrightarrow \pi_{r-1}(SO_{r+1}) \longrightarrow 0 .$$

The group $\pi_r(S^r)$ will be identified with the integers. Then $\partial(1) \in \pi_{r-1}(SO_r)$ is the element which corresponds to the tangent bundle of S^r. (See Steenrod [9, § 18]).

For each $\lambda \in \pi_r(SO_{r+1})$ let ξ denote the corresponding SO_{r+1}-bundle over S^{r+1}, and let $X(\xi)$ denote its Euler class (=top Stiefel-Whitney class with integer coefficients). Let μ be the standard generator of $H_{r+1}(S^{r+1}; Z)$. Then $f_*(\lambda)$ is equal to the negative of the " Euler number " $\langle X(\xi), \mu \rangle$. [Proof. Let $\mathfrak{o}(\xi) \in H^{r+1}(S^{r+1}; \pi_r(SO_{r+1}))$ denote the obstruction to the existence of a cross-section of ξ. Then $X(\xi)$, the obstruction to the existence of a cross-section in the associated sphere bundle, is equal to $f_*(\mathfrak{o}(\xi))$. According to Steenrod [9, p. 180] the identity $\langle \mathfrak{o}(\xi), \mu \rangle = -\lambda$ is satisfied. Therefore $\langle X(\xi), \mu \rangle = -f_*(\lambda)$].

Now if S^r is parallelizable then $\partial(1) = 0$, hence there exists $\lambda \in \pi_r(SO_{r+1})$ with $f_*(\lambda) = 1$. For the corresponding bundle ξ, the class $X(\xi)$ generates $H^{r+1}(S^{r+1}; Z)$; hence the class $w_{r+1}(\xi) = (X(\xi))_2$ is non-zero. Together with Theorem 1 this complete the proof.

It follows immediately that the real projective space P^r is parallelizable only for $r = 1, 3, 7$. (For consequences concerning the immersion of P^r in euclidean space see Milnor, Comm. Math. Helv. 30 (1956), p. 284).

COROLLARY 1. *There exists a division algebra of rank n over the real numbers only for $n = 1, 2, 4, 8$.*

PROOF. The existence of a bilinear product operation without zero-divisors in the vector space R^n implies that the projective space P^{n-1} is parallelizable. (See Stiefel [11, p. 216]). [ALTERNATIVE PROOF. Suppose that such a product operation in R^n is given. Then the correspondence $S^{n-1} \to GL_n$ defined by $x \to$ (left multiplication by x) gives rise to a GL_n-bundle ξ over S^n. It is not hard to verify that $w_n(\xi) \neq 0$].

COROLLARY 2. *For $r \geqq 8$ the groups $\pi_{r-1}(SO_r)$ are as follows:*

r modulo 8:	0	1	2	3	4	5	6	7
$\pi_{r-1}(SO_r)$:	$Z+Z$	*group of order* 4	$Z+Z_2$	Z_2	$Z+Z$	Z_2	Z	Z_2

PROOF. This follows from Bott's computation [2] of the stable groups $\pi_{r-1}(SO_{r+1})$, together with the exact homotopy sequence used to prove Theorem 2.

THEOREM 3. *Let M^{2n} be a simply-connected differentiable manifold such that the cohomology group $H^i(M^{2n}; Z)$ is infinite cyclic for $i = 0, n, 2n$, and zero otherwise. Then n must be 2, 4, or 8.*

(Examples are provided by the complex, quaternion, and Cayley projective planes. It will be shown in a later paper [7] that the condition of simple-connectivity can be eliminated. This will give an answer to Problem 5 of [5]).

PROOF. If α generates $H^n(M^{2n}; Z)$, then the Poincaré duality theorem implies that $\alpha \smile \alpha$ generates $H^{2n}(M^{2n}; Z)$. Hence

$$Sq^n : H^n(M^{2n}; Z_2) \longrightarrow H^{2n}(M^{2n}; Z_2)$$

in non-zero. The formulas of Wu [15] now imply that the Stiefel-Whitney class w_n of the tangent bundle θ is non-zero. Choose a map $g : S^n \to M^{2n}$ which, under the Hurewicz homomorphism, corresponds to a generator of $H_n(M^{2n}; Z)$. Then the bundle θ' over S^n induced from θ by g will satisfy $w_n(\theta') \neq 0$. Therefore n must be 1, 2, 4 or 8. Since the case $n = 1$ is easily excluded, this completes the proof.

Bott's theorem is related to the question of the existence of maps with Hopf invariant 1 as follows. Let $J : \pi_{n-1}(SO_m) \to \pi_{m+n-1}(S^m)$ be the homomorphism of G. W. Whitehead [13], and let $\gamma_n : \pi_{m+n-1}(S^m) \to Z_2$ be the generalized Hopf invariant of Steenrod [8], which is defined using the functional Sq^n operation. For each odd prime q let

$$\gamma_{q,i} : \pi_{m+2i(q-1)-1}(S^m) \longrightarrow Z_q$$

denote the corresponding homomorphism based on the reduced q^{th} power $\mathscr{P}^i$.

THEOREM 4a. *The image $J\pi_{n-1}(SO_m)$, $m \geqq n$, contains an element $J\lambda$ with generalized Hopf invariant $\gamma_n(J\lambda)$ different from zero only if n equals 2, 4, or 8.*

THEOREM 4b. *The image $J\pi_{2i(q-1)-1}(SO_m)$, $m \geqq 2i(q-1)$, contains an element $J\lambda$ with $\gamma_{q,i}(J\lambda)$ different from zero only if $i = 1$.*

PROOF OF 4b. Let ξ be the SO_m-bundle over S^n associated with λ, where $n = 2i(q-1)$. Let E be the total space of the associated bundle having the unit ball B^m as fibre, so that the boundary $\dot{E}$ is the total space of the associated sphere bundle. According to [7, Theorem 3, Corollary 1], the collapsed space $E/\dot{E}$ can be obtained from the sphere S^m by attaching an $(m+n)$-cell, using on attaching map in the homotopy class $J\lambda$. Thus the generalized Hopf invariant $\gamma_{q,i}(J\lambda)$ is non-zero if and only if the homomorphism

$$\mathscr{P}^i : H^m(E, \dot{E} ; Z_q) \longrightarrow H^{m+n}(E, \dot{E} ; Z_q)$$

is non-zero.

Let $\phi : H^j(S^n ; Z_q) \to H^{j+m}(E, \dot{E} ; Z_q)$ denote the isomorphism of Thom [12]. According to Wu [18, § IV] the class $\phi^{-1}\mathscr{P}^i\phi(1) \in H^n(S^n ; Z_q)$ can be expressed as a polynomial in the Pontrjagin classes of ξ, reduced modulo q. But these Pontrjagin classes are zero, except for $p_{i(q-1)/2}(\xi)$ which is divisible by $(i(q-1)-1)!$. For $i > 1$, since the number $(i(q-1)-1)!$ is divisible by q, it follows that the operation $\mathscr{P}^i$ must be zero.

Theorem 4a is proved in a similar way, using Theorem 1 together with Thom's definition of the Stiefel-Whitney classes. (See [12]).

Appendix

PROOF OF THE THEOREM OF WU. Following Hirzebruch [6] define the Pontrjagin class p_k of an O_m-bundle as $(-1)^k$ times the Chern class c_{2k} of the U_m-bundle induced by the inclusion $O_m \to U_m$. This is slightly different from the Pontrjagin class as defined by Pontrjagin and Wu. (Compare [17, Theorem 4]).

Consider the exact sequence of cohomology group corresponding to the coefficient sequence

$$0 \longrightarrow Z_2 \xrightarrow{\ i\ } Z_4 \xrightarrow{\ j\ } Z_2 \longrightarrow 0 ;$$

as well as the Pontrjagin squaring operation

$$\mathfrak{P} : H^{2k}(K ; Z_2) \longrightarrow H^{4k}(K ; Z_4) .$$

(See for example Whitehead [14]).

LEMMA 1. *The Pontrjagin class p_k of any O_m-bundle is related to the Stiefel-Whitney classes $w_1, \cdots, w_{4k}$ by an identity*

$$(p_k)_4 = \mathfrak{P}(w_{2k}) + i_* f_k(w_1; \cdots, w_{4k})$$

where f_k is a polynomial with coefficients in Z_2.

PROOF. It is clearly sufficient to consider the case of the universal bundle over the Grassmann space $G_m(R)$, with m large. The identity $j_*\mathfrak{P}(w) = w \smile w$ holds for any cohomology class w. Comparing this with the relation[3]

[3] See Wu [17, Theorem 3].

$$j_*((p_k)_4) = (p_k)_2 = w_{2k} \smile w_{2k}$$

it follows that

$$(p_k)_4 - \mathfrak{P} w_{2k} \in (\text{kernel}\, j_*) = i_* H^{4k}(G_m(R)\,;Z_2)\ .$$

Since the cohomology ring $H^*(G_m(R)\,;Z_2)$ is generated by the Stiefel-Whitney classes, this proves Lemma 1.

To prove the theorem it is only necessary to show that the coefficient of w_{4k} in f_k is non-zero. Let Υ denote the universal U_m-bundle over the complex Grassmann space $G_m(C)$. Recall that the cohomology ring $H^*(G_m(C)\,;Z)$ is a polynomial ring[4] generated by the Chern classes of Υ. The inclusion $U_m \to O_{2m}$ induces an O_{2m}-bundle over $G_m(C)$ which will be denoted by Υ_R. Applying Lemma 1 to this bundle Υ_R, the relations[5]

$$p_k(\Upsilon_R) = c_k(\Upsilon)^2 - 2c_{k-1}(\Upsilon)c_{k+1}(\Upsilon) + - \cdots \pm 2c_0(\Upsilon)c_{2k}(\Upsilon)$$

and

$$w_{2r+1}(\Upsilon_R) = 0\ , \qquad w_{2r}(\Upsilon_R) = (c_r(\Upsilon))_2$$

show that the polynomial f_k must satisfy

$$f_k(0, w_2, 0, w_4, \cdots, w_{4k}) = w_{2k-2}w_{2k+2} + w_{2k-4}w_{2k+4} + \cdots + w_0 w_{4k}\ .$$

Therefore $f_k(0, 0, \cdots, 0, w_{4k}) = w_{4k}$; which completes the proof.

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REFERENCES

1. A. BOREL and F. HIRZEBRUCH, *Characteristic classes and homogeneous spaces*, to appear (Amer. J. Math.).
2. R. BOTT, *On the stable homotopy of the classical groups*, Proc. Nat. Acad. Sci. U.S.A. 43 (1957), 933–935.
3. ———, *The space of loops on a Lie-group*, to appear (Mich. J. Math.).
4. S. S. CHERN, *Characteristic classes of Hermitian manifolds*, Ann. of Math. 47 (1946), 85–121.
5. F. HIRZEBRUCH, *Some problems on differentiable and complex manifolds*, Ann. of Math. 60 (1954), 213–236.
6. ———, Neue topologische Methoden in der algebraischen Geometrie, Springer, 1956.
7. J. MILNOR, *On spaces with a gap in cohomology*, to appear.
8. N. E. STEENROD, *Coholomogy invariants of mappings*, Ann. of Math. 50 (1949), 954–988.
9. ———, The topology of fibre bundles, Princeton, 1951.
10. ——— and J.H.C. WHITEHEAD, *Vector fields on the n-sphere*, Proc. Nat. Acad. Sci. U.S.A. 37 (1951), 58–63.
11. E. STIEFEL, *Über Richtungsfelder in den projektiven Räumen und einen Satz aus der reelen Algebra*, Comm. Math. Helv. 13 (1940), 201–218.
12. R. THOM, *Espaces fibrés en sphères et carrés de Steenrod*, Ann. sci. Ecole norm. sup 69 (1952), 109–182.

[4] Chern [4].

[5] See Hirzebruch [6, p. 68] and Steenrod [9, p. 212].

13. G. W. WHITEHEAD, *On the homotopy groups of spheres and rotation groups*, Ann. of Math. 43 (1942), 634–640.
14. J.H.C. WHITEHEAD, *On simply connected 4-dimensional polyhedra*, Comm. Math. Helv. 22 (1949), 48–92.
15. WU WEN-TSÜN, *Classes caractéristiques et i-carrés d'une variété*, C. R. Acad. Sci. Paris 230 (1950), 508–511.
16. ———, *Les i-carrés dans une variété grassmannienne*, C. R. Acad. Sci. Paris 230 (1950), 918–920.
17. ———, *On Pontrjagin classes I*, Scientia Sinica 3 (1954), 353–367.
18. ———, *On Pontrjagin classes II*, Scientia Sinica 4 (1955), 455–490.
19. ———, *On Pontrjagin classes III*, Acta Math. Sinica 4 (1954), 323–346.
20. R. BOTT and J. MILNOR, *On the parallelizability of the spheres*, Bull. Amer. Math. Soc. 64 (1958), 87–89.