1. (a) Recall from (14.4) in the lecture notes that a diffeomorphism \( f : Y \to Y \) of a closed manifold \( Y \) determines a bordism \( X_f \). Let \( f_0, f_1 \) be diffeomorphisms. Prove that \( X_{f_0} \) is diffeomorphic to \( X_{f_1} \) (as bordisms) if and only if \( f_0 \) is pseudoisotopic to \( f_1 \).

(b) Find a manifold \( Y \) and diffeomorphisms \( f_0, f_1 : Y \to Y \) which are pseudoisotopic but not isotopic.

2. (a) Let \( S \) be a set with composition laws \( \circ_1, \circ_2 : S \times S \to S \) and distinguished element \( 1 \in S \). Assume (i) \( 1 \) is an identity for both \( \circ_1 \) and \( \circ_2 \); and (ii) for all \( s_1, s_2, s_3, s_4 \in S \) we have

\[
(s_1 \circ_1 s_2) \circ_2 (s_3 \circ_1 s_4) = (s_1 \circ_2 s_3) \circ_1 (s_2 \circ_2 s_4).
\]

Prove that \( \circ_1 = \circ_2 \) and that this common operation is commutative and associative.

(b) Let \( C \) be a symmetric monoidal category. Apply (a) to \( C(1,1) \), where \( 1 \in C \) is the tensor unit.

3. Let \( y \in C \) be a dualizable object in a symmetric monoidal category, and suppose \( (y^!, c, e) \) and \( (\tilde{y}^!, \tilde{c}, \tilde{e}) \) are two sets of duality data. Prove there is a unique map \( (y^!, c, e) \to (\tilde{y}^!, \tilde{c}, \tilde{e}) \).

4. For each of the following symmetric monoidal categories determine all of the dualizable objects.
   (a) \((\text{Top}, \amalg)\), the category of topological spaces and continuous maps under disjoint union.
   (b) \((\text{Ab}, \oplus)\), the category of abelian groups and homomorphisms under direct sum.
   (c) \((\text{Mod}_R, \otimes)\), the category of \(R\)-modules and homomorphisms under tensor product, where \( R \) is a commutative ring.
   (d) \((\text{Set}, \times)\), the category of sets and functions under Cartesian product.

5. Recall that every category \( C \) has an associated groupoid \(|C|\) obtained from \( C \) by inverting all of the arrows. What is \(|\text{Bord}_{(1,2)}|\)? \(|\text{Bord}^{\text{Spin}}_{(1,2)}|\)? What are all \(\text{Vect}_C\)-valued invertible topological quantum field theories with domain \(\text{Bord}_{(1,2)}\)? \(\text{Bord}^{\text{Spin}}_{(1,2)}\)?

6. Fix a finite group \( G \). Let \( C \) denote the groupoid \( G//G \) of \( G \) acting on itself by conjugation. Let \( D \) denote the groupoid of principal \( G \)-bundles over \( S^1 \). (A principal \( G \)-bundle is a regular, or Galois, cover with group \( G \).) Prove that \( C \) and \( D \) are equivalent groupoids. You should spell out precisely what these groupoids are.
7. Explain why each of the following fails to be a natural map \( \eta: F \to G \) of symmetric monoidal functors \( F, G: C \to D \).

(a) \( F, G \) are the identity functor on \( \text{Vect}_k \) for some field \( k \), and \( \eta(V): V \to V \) is multiplication by 2 for each vector space \( V \).

(b) \( C, D \) are the category of algebras over a field \( k \), the functor \( F \) maps \( A \mapsto A \otimes A \), the functor \( G \) is the identity, and \( \eta(A): A \otimes A \to A \) is multiplication.

8. In this problem you construct a simple TQFT \( F: \text{Bord}_{(0,1)} \to \text{Vect}_\mathbb{Q} \). For any manifold \( M \) let \( \mathcal{C}(M) \) denote the groupoid of principal \( G \)-bundles over \( M \), as in Problem 6.

(a) For a compact 0-manifold \( Y \), define \( F(Y) \) as the vector space of functions \( \mathcal{C}(Y) \to \mathbb{Q} \). Say what you mean by such functors on a groupoid.

(b) For a closed 1-manifold \( X \) define

\[
F(X) = \sum_{[P] \in \pi_0 \mathcal{C}(X)} \frac{1}{\# \text{Aut}(P)},
\]

where the sum is over equivalence classes of principal \( G \)-bundles. Extend this to all bordisms \( X: Y_0 \to Y_1 \).

(c) Check that \( F \) is a symmetric monoidal functor.

(d) Calculate \( F \) on a set of duality data for the point \( \text{pt} \in \text{Bord}_{(0,1)} \). Use it to compute \( F(S^1) \).

9. Fix a nonzero number \( \lambda \in \mathbb{C} \). Construct an invertible TQFT \( F: \text{Bord}^{SO}_{(1,2)} \to \text{Vect}_\mathbb{C} \) such that for a closed 2-manifold \( X \) we have \( F(X) = \lambda^{\chi(X)} \), where \( \chi(X) \) is the Euler characteristic. Can you extend to the bordism category \( \text{Bord}^{SO}_{(1,2)} \)?