Problem Set # 4
M392C: Bordism Old and New
Due: December 6, 2012

1. The embedding \( U(m) \hookrightarrow O(2m) \) of the unitary group into the orthogonal group determines a \( 2m \)-dimensional tangential structure \( BU(m) \to BO(2m) \). Compute the integral homology \( H_\bullet(MTU(m)) \) of the associated Madsen-Tillmann spectrum.

2. For each of the following maps \( \mathcal{F}: \text{Man}^{\text{op}} \to \text{Set} \), answer: Is \( \mathcal{F} \) a presheaf? Is \( \mathcal{F} \) a sheaf?
   (a) \( \mathcal{F}(M) = \) the set of smooth vector fields on \( M \)
   (b) \( \mathcal{F}(M) = \) the set of orientations of \( M \)
   (c) \( \mathcal{F}(M) = \) the set of sections of \( \text{Sym}^2 T^* M \)
   (d) \( \mathcal{F}(M) = \) the set of Riemannian metrics on \( M \)
   (e) \( \mathcal{F}(M) = \) the set of isomorphism classes of double covers of \( M \)
   (f) \( \mathcal{F}(M) = H^q(M; A) \) for some \( q \geq 0 \) and abelian group \( A \)

3. Define a sheaf \( \mathcal{F} \) of categories on \( \text{Man} \) which assigns to each test manifold \( M \) a groupoid of double covers of \( M \). Be sure to check that you obtain a presheaf—compositions map to compositions—which satisfies the sheaf condition. Describe \( |\mathcal{F}| \) and \( B|\mathcal{F}| \). Compute the set \( \mathcal{F}[M] \) of concordance classes of double covers on \( M \).

4. (a) Fix \( q \geq 0 \). Define a sheaf \( \mathcal{F} \) of sets on \( \text{Man} \) which assigns to each test manifold \( M \) the set of closed differential \( q \)-forms. Compute \( \mathcal{F}[M] \). Identify \( |\mathcal{F}| \).
   (b) Fix \( k > 0 \). Fix a complex Hilbert space \( \mathcal{H} \). Define a sheaf \( \mathcal{F} \) of sets on \( \text{Man} \) which assigns to each test manifold \( M \) the set of rank \( k \) vector bundles \( \pi: E \to M \) together with an embedding \( E \hookrightarrow M \times \mathcal{H} \) into the vector bundle with constant fiber \( \mathcal{H} \) and a flat covariant derivative operator. (The flat structure and embedding are uncorrelated.) Discuss briefly why \( \mathcal{F} \) is a sheaf. Compute \( \mathcal{F}[M] \). Identify \( |\mathcal{F}| \).