Problem Set # 2
M365C: Real Analysis I
Due: January 28

Problems in Rudin
Chapter 1 (page 21): 1, 4, 5, 7, 16, 17

Other Problems

1. Show that the rational numbers \( \mathbb{Q} \) form an ordered field which satisfies the Archimedean property. What property of the real numbers \( \mathbb{R} \) is not satisfied by \( \mathbb{Q} \)?

2. A real polynomial \( p \) has the form \( p(x) = p_n x^n + p_{n-1} x^{n-1} + \cdots + p_0 \) for real numbers \( p_0, p_1, \ldots, p_n \) and some nonnegative integer \( n \). Unless \( p \) is the zero polynomial we assume \( p_n \neq 0 \) and say \( p \) has degree \( n \). A real rational function \( p/q \) is the ratio of two polynomials (of possibly different degrees) where \( q \) is not identically zero. Let \( F \) be the set of real rational functions.
   
   (a) Give \( F \) the structure of a field. What are the operations of addition and multiplication? What are the additive and multiplicative identity elements? Verify the field axioms.
   
   (b) Define \( p/q \in F \) to be positive \( (p/q > 0) \) if

   \[
   \frac{p}{q}(x) = \frac{p(x)}{q(x)} = \frac{p_n x^n + \cdots + p_0}{q_m x^m + \cdots + q_0}
   \]

   satisfies \( p_n/q_m > 0 \). Why is this well-defined? Then for \( f, g \in F \) we define \( f > g \) if and only if \( f - g \) is positive. Prove that with this definition \( F \) is an ordered field. (Hint: in particular, you must prove that \( F \) is an ordered set.)

   (c) Does \( F \) satisfy the Archimedean property? Proof or counterexample.

3. Consider the function

   \[
   f: (-1, 1) \to \mathbb{R}
   \]

   \[
   x \mapsto x^2
   \]

   What is the domain of \( f \)? What is the codomain? What is the range? Is \( f \) injective? Surjective? Bijective? What is \( f([-1/3, 1/2]) \)? What is \( f^{-1}(\{3, 4, 5\}) \)? How does the cardinality of \( f^{-1}(y) \) depend on \( y \in \mathbb{R} \)? If \( f(1) \) defined?