Problem Set # 6

M392C: K-theory

Here are several problems about Clifford algebras and Clifford modules.

1. (a) Prove that there are no nontrivial graded ideals in $C\ell_n$.

(b) Let $G$ be the extra-special 2-group in $C\ell_n$ generated by $\{\pm e_1, \ldots, \pm e_n\}$. (It is a finite group.) Describe its representation theory. How are its representations related to Clifford modules?

(c) If $n$ is even we proved that $C\ell_n^C$ is a super matrix algebra. What can you say if $n$ is odd?

2. (a) Prove that the opposite of $C\ell(V,Q)$ is isomorphic to $C\ell(V,-Q)$.

(b) Construct an isomorphism of the spin groups which sit inside $C\ell_n$ and $C\ell_{-n}$. Is there an isomorphism of pin groups?

(c) Prove that the category of modules over $C\ell_{-4}$ is isomorphic to the category of quaternionic vector spaces.

3. (a) Construct isomorphisms of ungraded algebras

\[ C\ell_{n+2} \rightarrow C\ell_{-n} \otimes C\ell_2 \]
\[ C\ell_{-(n+2)} \rightarrow C\ell_n \otimes C\ell_{-2} \]

The tensor product is the tensor product of ungraded algebras.

(b) Write the underlying ungraded algebras of $C\ell_5, C\ell_{-5}$ as (sums of) matrix algebras.

4. In this problem we use only the ‘negative’ Clifford algebras. For $n \geq 0$ define $A_{-n}$ as the Grothen-dieck group, or abelian group completion, of the group of isomorphism classes of $C\ell_{-n}$-modules modulo restrictions of $C\ell_{-(n+1)}$-modules.

(a) Compute $A_{-n}$ for all $n$. What are generators?

(b) Define a ring structure on $A = \bigoplus_{n=0}^{\infty} A_{-n}$. Find generators and relations.

(c) Treat a complex version.

(d) How is this related to the Atiyah-Bott-Shapiro construction? First, if a $C\ell_{-n}$-module is the restriction of a $C\ell_{-(n+1)}$-module, what can you say about the vector bundle over the sphere constructed in (6.64)?
5. In this problem you construct the canonical Dirac operator on an oriented spin Riemannian manifold \( M \) of dimension \( n \).

(a) Forget ‘spin’ for now and construct \( SO(M) \to M \) the bundle of oriented orthonormal frames as a principal \( SO_n \)-bundle. Define a spin structure to be a lift to a principal \( Spin_n \)-bundle. Explain the meaning of ‘lift’. Assume now that \( M \) is spin.

(b) Construct the associated bundle whose fiber is the Clifford algebra \( C\ell_n \) by letting \( Spin_n \) act by left multiplication. It is a \( \mathbb{Z}/2\mathbb{Z} \)-graded bundle \( S \to M \) with a right \( C\ell_n \)-action. Show that it is equivalent to a left \( C\ell_{-n} \)-action.

(c) Use the Levi-Civita connection to induce a covariant derivative on sections of \( S \to M \). Compose with Clifford multiplication to construct the Dirac operator. Show it is odd skew-adjoint and commutes with the \( C\ell_{-n} \)-action.

6. (a) Let \( H \) be a Lie group and \( P \to X \) a principal \( H \)-bundle. Suppose \( \rho: H \to G \) is a homomorphism of Lie groups. Construct an associated principal \( G \)-bundle \( P \times_H G \to X \).

(b) Now suppose \( Q \to X \) is a principal \( G \)-bundle. A reduction to \( H \) is a pair \( (P, \theta) \) of a principal \( H \)-bundle \( P \to X \) and an isomorphism \( \theta: Q \to P \times_H G \) of principal \( G \)-bundles. Show that if \( \rho \) is an inclusion, then isomorphism classes of reductions (define) are in 1:1 correspondence with sections of \( P/H \to X \).

(c) Explain part (b) for \( X \) a smooth \( n \)-manifold, \( Q \to X \) its bundle of frames, and \( \rho: H \to G \) the inclusion \( O_n \hookrightarrow GL_n\mathbb{R} \).

(d) Investigate the existence and uniqueness question for reductions in two cases: (i) \( \rho \) is the inclusion of a subgroup of order 2; (ii) \( \rho \) is a 2:1 cover of Lie groups.