Problem Set # 7
M392C: K-theory

1. Suppose \( p: E \rightarrow B \) is a fibration. Assume \( E, B \) have basepoints \( e, b \). For \( b' \in B \) let \( P_e((E; p^{-1}(b'))) \) denote the space of paths in \( E \) which begin at \( e \) and terminate on the fiber \( p^{-1}(b') \). Prove that \( p \) induces a fibration

\[
P_e((E; p^{-1}(b'))) \rightarrow P_b(B; b')
\]

with contractible fibers. What assumptions do you need to make on the topological spaces \( E, B \)? Conclude that \( p \) is a weak homotopy equivalence. When can you conclude that \( p \) is a homotopy equivalence?

2. Fix a positive integer \( n \). Let \( E \) denote the space of skew-Hermitian \( n \times n \) matrices with operator norm \( \leq 1 \). (The eigenvalues \( i\lambda_1, \ldots, i\lambda_n \) satisfy \( |\lambda_j| \leq 1 \).) Consider the exponential map

\[
p: E \rightarrow U(n)
A \mapsto \exp(\pi A)
\]

(a) For each \( k \) between 0 and \( n \) prove that the restriction of \( p \) over the subspace of \( U(n) \) consisting of unitary matrices with \((-1)\)-eigenspace of dimension \( k \) is a fiber bundle. What is the fiber?

(b) Show that \( p \) is a quasifibration.

3. Use the contractibility of the unit sphere in Hilbert space, proved in a previous problem set, to prove that the infinite dimensional Stiefel manifold is contractible.

4. Go through the proof of Kuiper’s theorem and prove that all homotopies are continuous.