Problem Set # 9
M392C: K-theory

Please write up the solutions to 6 problems and turn in by Thursday, November 5.

In this problem set $G$ is a compact Lie group and $\mathfrak{g}$ its Lie algebra.

1. The Killing form $\kappa: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ is defined for any Lie algebra $\mathfrak{g}$ by $\kappa(\xi_1, \xi_2) = \text{Trace}(\text{ad}(\xi_1) \circ \text{ad}(\xi_2))$. In this problem assume $\mathfrak{g}$ is the Lie algebra of a compact Lie group.

(a) Prove that the Killing form is negative semi-definite.

(b) Prove that the Ad-action of $G$ on $\mathfrak{g}$ is orthogonal for the Killing form.

(c) Assume the Killing form induces a bi-invariant metric on $G$. Prove that, in fact, for any bi-invariant metric the Riemannian exponential map at the identity agrees with the exponential map defined from the Lie group structure.

2. Suppose $G$ is a connected compact Lie group.

(a) Let $\Omega^\bullet_{\text{left}}(G) \subset \Omega^\bullet(G)$ denote the vector subspace of left-invariant differential forms. Show that $\Omega^\bullet_{\text{left}}(G)$ is in fact a sub-differential graded algebra, i.e., it is closed under multiplication and the differential $d$.

(b) Construct an isomorphism

\[
\bigwedge^\bullet \mathfrak{g}^* \rightarrow \Omega^\bullet_{\text{left}}(G).
\]

Transfer the differential on $\Omega^\bullet_{\text{left}}(G)$ to $\bigwedge^\bullet \mathfrak{g}^*$ and write a formula for it. In this way you obtain a differential graded complex defined directly from the Lie algebra $\mathfrak{g}$. Observe that your definition works for any Lie algebra (it needn’t be the Lie algebra of a compact Lie group).

(c) Prove that the inclusion in part (a) induces an isomorphism on cohomology. A map of cochain complexes with this property is called a quasi-isomorphism. So you can compute the de Rham cohomology of $G$ from this Lie algebra complex. (Hint: Average over $G$ to construct a left-invariant form from an arbitrary form.)

(d) Use the inverse map $g \mapsto g^{-1}$ to show that the differential of a bi-invariant differential form vanishes. Show that the de Rham cohomology of $G$ is isomorphic to the algebra of bi-invariant forms.

(e) Use these ideas to compute $H^\bullet_{\text{dR}}(SU_2)$.

(f) Endow $G$ with a bi-invariant metric. Is there a relationship between harmonic forms and bi-invariant forms?
1. (a) Consider the adjoint action of $U_n$. Let $T \subset U_n$ be the maximal torus of diagonal matrices. What are the root spaces and the roots?

(b) Repeat for $SU_n$.

(c) Repeat for $SO_n$ and $Sp_n$.

4. Consider the group $SU_3$ of $3 \times 3$ unitary matrices of determinant one.

(a) Compute the Lie algebra $su_3$ of $SU_3$. What is $\dim SU_3$?

(b) Construct an $Ad$-invariant bilinear form on $su_3$.

(c) Choose a maximal torus $T \subset SU_3$ to be the diagonal matrices. What is the rank of $SU_3$?

Identify the lattices $\Pi$ and $\Lambda$ as subsets of $t$ and $t^*$ respectively, where $t = \text{Lie}(T)$.

(d) Find the normalizer $N(T)$ to the torus. Identify the Weyl group $W = N(T)/T$.

(e) Restrict the adjoint representation of $SU_3$ to $T$. Diagonalize this action by complexifying the Lie algebra and compute the function $\lambda \mapsto \mathbb{Z}$ which specifies the multiplicities of the weights. These are the roots of $SU_3$.

(f) Compute the weights of the standard representation of $SU_3$ on $\mathbb{C}^3$.

(g) Compute the weights of the symmetric square of the standard representation.

5. Let $G$ be a compact Lie group. It is true that there is a countable set of isomorphism classes of irreducible complex representations. Let $\{V_i\}$ be a choice of a set of representative irreducible representations. For any finite dimensional representation $V$ construct a canonical isomorphism

$$\bigoplus_i \text{Hom}_G(V_i, V) \otimes V_i \rightarrow V.$$ 

You might even consider the meaning of ‘canonical’ and prove that your isomorphism is just that.

6. (a) Let $V$ be a complex vector space. Define the complex conjugate space $\overline{V}$ to be equal to $V$ as an abelian group and with scalar multiplication complex conjugate to that in $V$. In other words, if $v \in V$ equals $\overline{v} \in \overline{V}$ (recall that $V = \overline{V}$ as a set, even as an abelian group), then for any complex number $c$, we have $\overline{c \cdot v} = \overline{c} \cdot \overline{v}$. Here the first ‘·’ is scalar multiplication in $V$, the second in $\overline{V}$.

(b) A real structure on $V$ is a linear map $J \colon V \to \overline{V}$ which satisfies $J \circ J = \text{id}_V$. Show that the fixed points of $J$ form a real vector space $W$. Produce a canonical isomorphism $W \otimes \mathbb{C} \rightarrow V$.

(c) A quaternionic structure on $V$ is a linear map $J \colon V \to \overline{V}$ which satisfies $J \circ J = -\text{id}_V$. Show that in this case $V$ is naturally a module over the quaternions $\mathbb{H}$. It is often convenient to treat quaternionic vector spaces as complex vector spaces with this extra structure.
(d) Suppose $G$ is a compact Lie group and $V$ a complex representation, i.e., a complex vector space with a linear $G$-action. Then $V$ is self-conjugate if there is a real or quaternionic structure which is preserved by the group $G$. Give an example of a self-conjugate representation. Give an example of a representation which is not self-conjugate. Show that the tensor product of self-conjugate representations is self-conjugate. Discuss the various cases: real $\otimes$ real, real $\otimes$ quaternionic, etc.

(e) Explore how to construct a real representation from a complex representation, a complex representation from a real one, a quaternionic representation from a complex representation, etc. You will be defining certain functors between categories of representations. Spell it out in that language. Investigate various compositions of your functors.

7. Let $G$ be a compact Lie group and $V$ a complex representation. We proved in lecture that $V$ carries an invariant hermitian form, even one which is positive definite. Now investigate the existence of an invariant bilinear form. Give examples to demonstrate existence or non-existence. If $V$ is irreducible show that the space of invariant forms is zero or one-dimensional, and in the latter case all nonzero forms are either symmetric or skew-symmetric. How does the existence of invariant forms relate to the self-conjugacy of the representation?

8. True or false. Proof or example.
   (a) There is a nontrivial homomorphism $SO_3 \to Sp_1$.
   (b) There is a nontrivial homomorphism $Sp_1 \to SO_3$.
   (c) There is a nontrivial homomorphism $SO_5 \to Sp_2$.
   (d) There is a nontrivial homomorphism $Sp_2 \to SO_5$.
   (e) There is a nontrivial homomorphism $Sp_2 \to Spin_5$.
   (f) There is a nontrivial homomorphism $SU_3 \to SO_7$.
   (g) There is a nontrivial homomorphism $SO_7 \to SU_3$.
   (h) There is a nontrivial homomorphism $Spin_7 \to Spin_8$.

9. Apply the Weyl character formula to deduce the characters of the representations discussed in Problem 4. Explore the rank 2 groups $SO_4$, $SO_5$, $Sp_2$ and $U_2$. Learn the graphic algorithm at the end of the article of Bott (see web page) for computing the character. There is graph paper available on the web page for $SU_3$. I welcome graph paper for other groups!