Problem Set # 10
M392C: Riemannian Geometry

1. Let $X$ be a complex manifold and $P \to X$ a holomorphic principal $GL_m \mathbb{C}$-bundle. Show that a $GL_m \mathbb{C}$-invariant horizontal distribution $H \subset TP$ is invariant under the complex structure $I$ if and only if the corresponding connection form $\Theta \in \Omega^1(gl_m \mathbb{C})$ is of type $(1,0)$.

2. Let $X$ be a smooth manifold with an almost complex structure, i.e., a section $I$ of $\text{End}(TX) \to X$ which satisfies $I^2 = -\text{id}_{TX}$. The $+i$-eigenspace of $I$ is a complex distribution $T(1,0)X \subset TX \otimes \mathbb{C}$. Its Frobenius tensor $\Phi$ is a $(2,0)$-form with values in $T(0,1)X = T(1,0)X$. Assume that $\Phi = 0$. Let $\nabla$ be a torsionfree covariant derivative on $TX \to X$. (Why does one exist?) For (real) vector fields $\xi, \eta$ set

$$
\nabla' \xi \eta = \nabla \xi \eta - (\nabla_I \eta) I \xi - I((\nabla_I \eta) \xi) - 2I((\nabla_I \xi) \eta).
$$

Prove that $\nabla'$ is a torsionfree covariant derivative which satisfies $\nabla' I = 0$.

3. Let $X^{2m}$ be a complex manifold with complex structure $I$ and suppose $\langle -,- \rangle$ is a Riemannian metric on $X$. Assume $I$ is orthogonal. Let $\omega$ be the associated 2-form. Here are two more proofs that $d\omega = 0$ implies $X$ is Kähler.

(a) Show that $\omega$ has type $(1,1)$.

(b) Let $\nabla$ denote the Levi-Civita covariant derivative on $TX \to X$. Show that for vector fields $\xi, \eta, \zeta$ we have

$$
2\langle (\nabla I) \eta, \zeta \rangle = 3d\omega(\xi, I\eta, I\zeta) - 3d\omega(\xi, \eta, \zeta).
$$

Conclude that $X$ is Kähler if $d\omega = 0$.

(c) Let $\theta^1, \ldots, \theta^{2m}$ be a local orthonormal coframing adapted to $I$: the dual framing $\xi_1, \ldots, \xi_{2m}$ satisfies $\xi_2 = I\xi_1$, $\xi_4 = I\xi_3$, etc. Show that

$$
\omega = \theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4 + \ldots.
$$

Let $\Theta^i_j$ denote the Levi-Civita connection forms, characterized by the equations

$$
d\theta^i + \Theta^i_j \wedge \theta^j = 0
$$

$$
\Theta^i_j + \Theta^j_i = 0
$$

Show that $I$ is parallel if and only if $\Theta I = I\Theta$ if and only if

$$
\Theta^3_2 + \Theta^4_1 = 0
$$

$$
\Theta^1_3 - \Theta^2_4 = 0
$$

$$
\ldots
$$

Prove that this is satisfied if and only if $d\omega = 0$. 
4. (a) Let $V$ be an $m$-dimensional complex vector space with a Hermitian metric. Its automorphism group is denoted $U(V)$. Let $SU(V)$ denote the closed Lie subgroup of automorphisms which act trivially on $\bigwedge^m V^*$. Show that these are precisely the automorphisms of determinant one. An element of $\bigwedge^m V^*$ is a complex volume form on $V$: it attaches a complex number to the complex parallelepiped spanned by $m$ (ordered) vectors in $V$.

(b) Let $X^{2m}$ be a Riemannian manifold whose holonomy group is a subgroup of $SU_m$. Construct a nonzero parallel complex volume form $\Omega \in \Omega_X^{m,0}$. Prove that $d\Omega = 0$. 