I have been posting notes and handouts on the website, so be sure to check often.

**Problems**

1. Let $A$ be a $\mathbb{Z}$-graded algebra and $T_1, T_2$ derivations of degree $s_1, s_2$, respectively. Prove that the commutator $[T_1, T_2]$ is a derivation of degree $s_1 + s_2$. Use the Koszul sign rule throughout.

2. (a) Suppose $A$ is an $n \times n$ real matrix. Define $e^A = \exp(A)$ using a power series. Prove carefully that the series does define a matrix.
   
   (b) Prove that $e^{A+B} = e^A e^B$ if $A$ and $B$ commute. In particular, show that $e^A$ is invertible. What is the first correction to this formula if $A$ and $B$ do not commute?
   
   (c) Compute the derivative of $e^{tA}$ with respect to the real variable $t$.
   
   (d) What can you say about $\det e^A$? What can you say about $e^A$ if $A$ is skew-symmetric?

3. Let $\omega$ be a $(k-1)$-form on a manifold $M$ and $\xi_1, \ldots, \xi_k$ vector fields on $M$. Compute $d\omega(\xi_1, \ldots, \xi_k)$. (In lecture we covered the case $k = 2$.)

4. (a) State carefully what it means for a Lie group $G$ to act on a manifold $M$ on the left or on the right.
   
   (b) If $G$ acts on $M$, then there is an induced linear map $\mathfrak{g} \to \mathcal{X}(M)$ from the Lie algebra of $G$ to the linear space of vector fields on $M$. Show that for a right action the map $\mathfrak{g} \to \mathcal{X}(M)$ is a homomorphism of Lie algebras. For a right action it is an antihomomorphism: the bracket of the image is minus the image of the bracket.

5. (a) Let $G$ be a Lie group. The left-invariant forms are closed under $d$, so form a subcomplex of the de Rham complex of $G$. As a vector space identify the left-invariant forms as the exterior algebra $\wedge^* \mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of $G$. Construct the de Rham differential $d$ on $\wedge^* \mathfrak{g}^*$ in terms of the Lie bracket.
   
   (b) Suppose $G$ is a compact Lie group and $\alpha$ a bi-invariant form. (In other words, $\alpha$ is both left-invariant and right-invariant.) Prove that $d\alpha = 0$.
   
   (c) Compute the complex of left-invariant forms and bi-invariant forms for the circle group $\mathbb{T}$ (consisting of complex numbers of unit norm) and for the group $SU_2$. What happens for $SL_2(\mathbb{R})$, the group of $2 \times 2$ real matrices of determinant one?
6. Let \( G \) be a Lie group. A **torsor** for \( G \) is a smooth manifold \( T \) on which \( G \) acts simply transitively. Thus a **right \( G \)-torsor** is a manifold \( T \) with a right \( G \) action \( T \times G \to T \) so that the map \( T \times G \to T \times T \) defined by \((t, g) \mapsto (t, t \cdot g)\) is a diffeomorphism.

(a) Let \( L \) be a real inner product space of dimension one. Prove that the elements of unit norm in \( L \) form a torsor for \( \mathbb{Z}/2\mathbb{Z} \).

(b) Let \( V \) be a real vector space and \( \mathcal{B}(V) \) the space of all ordered bases of \( V \). It is convenient to regard a basis of \( V \) as an invertible linear map \( b : \mathbb{R}^n \to V \). Then the group \( GL_n(\mathbb{R}) \) of invertible linear maps \( g : \mathbb{R}^n \to \mathbb{R}^n \) acts on the right by composition. Prove that \( \mathcal{B}(V) \) is a right \( GL_n(\mathbb{R}) \)-torsor.

(c) Now endow \( V \) with an inner product and show that the space \( \mathcal{O}(V) \) of orthonormal bases is a right \( O_n \)-torsor. What if we endow \( V \) with an orientation instead of a metric? What if we consider an oriented inner product space?

(d) Let \( E \) be a Euclidean space and \( \mathcal{O}(E) \) the space of all orthonormal frames at all possible points of \( E \). Here an orthonormal frame is an isometry \( f : \mathbb{E}^n \to E \) from the standard Euclidean space to \( E \). Construct a right action of the Euclidean group \( \text{Euc}_n \) and show that \( \mathcal{O}(E) \) is a right \( \text{Euc}_n \)-torsor.

(e) Verify that the canonical left-invariant 1-form on a Lie group \( G \) is well-defined on a right \( G \)-torsor (but it is not right-invariant). Show that it satisfies the Maurer-Cartan equation.

7. Example or proof of nonexistence: A codimension 1 foliation on the sphere \( S^4 \).