I have been posting notes and handouts on the website, so be sure to check often.

Problems

1. Let $G$ be a smooth manifold whose underlying set is equipped with a group structure, and assume that multiplication $m: G \times G \to G$ is smooth. Prove that the inverse map $i: G \to G$ is also smooth, and hence $G$ is a Lie group.

2. Let $G$ be a Lie group. Recall that $\text{Ad}: G \to \text{Aut}(\mathfrak{g})$ is defined by differentiating conjugation. Namely, if for $g \in G$ we define $A_g: G \to G$ by $A_g(x) = gxg^{-1}$, then $\text{Ad}_g = d(A_g)_e$.

   (a) Prove that $\text{Ad}_g$ is an automorphism of the Lie algebra $\mathfrak{g}$, i.e., it preserves the Lie bracket:

   $$\text{Ad}_g [\xi, \eta] = [\text{Ad}_g \xi, \text{Ad}_g \eta], \quad \xi, \eta \in \mathfrak{g}.$$  

   (b) Compute $d(\text{Ad}_g)_e$ in terms of the Lie bracket.

3. In the last problem set you learned that the set of bases of a vector space $V$ is a right torsor for $GL_n(\mathbb{R})$. Here are other geometric examples of torsors—verify that they are indeed torsors.

   (a) If $M$ is an orientable manifold, then the set of orientations is a torsor for $H^0(M; \mathbb{Z}/2\mathbb{Z})$, the group of locally constant functions $M \to \mathbb{Z}/2\mathbb{Z}$.

   (b) (This requires that you know about spin structures.) If $M$ is a spinable manifold (with a fixed orientation), then the set of equivalence classes of spin structures compatible with the given orientation is a torsor for $H^1(M; \mathbb{Z}/2\mathbb{Z})$.

   (c) If $\bar{a} \in \mathbb{R}/\mathbb{Z}$, then $\{x \in \mathbb{R} : x \equiv \bar{a} \pmod{1}\}$ is a $\mathbb{Z}$-torsor.

   (d) The fiber of a regular covering space $\tilde{X} \to X$ is a torsor for the group of deck transformations.

   (e) An affine space $A$ is a torsor for its underlying vector space of translations $V$.

4. Let $G$ be a Lie group and $\theta$ the (left-invariant) Maurer-Cartan form.

   (a) Compute $m^* \theta$, where $m: G \times G \to G$ is the multiplication map.

   (b) Compute $i^* \theta$, where $i: G \to T$ is inversion.
5. (a) What is the space of left-invariant metrics on a Lie group $G$? What is the space of bi-invariant metrics?

(b) Let $SU_2$ be the group of all $2 \times 2$ hermitian matrices with determinant one. (The entries are complex numbers.) Show that $SU_2$ is a Lie group. What is its dimension? Is it compact? Find all bi-invariant metrics on $SU_2$.

(c) Let $SL_2(\mathbb{R})$ be the group of $2 \times 2$ real matrices with determinant one. Prove that $SL_2(\mathbb{R})$ is a Lie group. Find all bi-invariant metrics on $SU_2$.

6. Suppose $N$ is a manifold and $G$ a Lie group. Let $\theta^i$, $i = 1, \ldots, n$ be a basis of left-invariant 1-forms on $G$ and suppose
\[ d\theta^i + \frac{1}{2} c_{jk}^i \theta^j \wedge \theta^k = 0 \]
for constants $c_{jk}^i$. Let $\theta^i_N$, $i = 1, \ldots, n$ be 1-forms on $N$. Consider the ideal of differential forms on $N \times G$ generated by $\pi^*_N \theta^i - \pi^*_G \theta^i$, where $\pi_1: N \times G \to N$ and $\pi_2: N \times G \to G$ are projections. Prove that this ideal is closed under $d$ if and only if
\[ d\theta^i_N + \frac{1}{2} c_{jk}^i \theta^j_N \wedge \theta^k_N = 0 \]

7. Let $E$ be a 3-dimensional Euclidean space and $\gamma: (a, b) \to E$ a smooth map such that $|\dot{\gamma}| = 1$. In other words, $\gamma$ is the unit speed parametrization of a curve. Assume further that the acceleration $\ddot{\gamma}$ is nowhere zero. Finally, assume the normal bundle to (the image of) $\gamma$ is oriented.

(a) Construct a canonical lift $\tilde{\gamma}: (a, b) \to \mathcal{B}_O(E)$ to the orthonormal frame bundle of $E$. This is called the Frenet frame. In other words, construct a curve $(e_1(t), e_2(t), e_3(t))$ of orthonormal frames of the vector space of translations of $E$. Make this construction so that $e_1$ is tangent to the curve, and $e_2$ is determined by the acceleration.

(b) Compute the pullbacks of the Maurer-Cartan forms $\theta^i, \Theta^i_j$ on $\mathcal{B}_O(E)$. You will meet two functions of $t$ called curvature and torsion.

(c) Given curvature and torsion functions on an interval $(a, b)$, prove that there exists a curve in $E$, unique up to a Euclidean motion, with the given curvature and torsion. Are there any restrictions on the curvature and torsion?

(d) Compute curvature and torsion for a plane curve. For a helix.

8. Let $X$ be a Riemannian manifold and $\xi \in \mathcal{X}(X)$ a vector field on $X$. The goal is to define an operator
\[ \nabla_{\xi}: \mathcal{X}(X) \to \mathcal{X}(X) \]
which satisfies the following two properties for all \( \eta, \zeta \in \mathcal{X}(X) \):

\[
\xi \langle \eta, \zeta \rangle = \langle \nabla_\xi \eta, \zeta \rangle + \langle \eta, \nabla_\xi \zeta \rangle \\
\nabla_\xi \eta - \nabla_\eta \xi = [\xi, \eta].
\]

(a) Use these properties to derive a formula for \( \langle \nabla_\xi \eta, \zeta \rangle \).

(b) Prove that \( \nabla_\xi \eta \) is linear over functions (tensorial) in \( \xi \) and satisfies a Leibniz rule in \( \eta \).

(c) Let \( x^1, \ldots, x^n \) be local coordinates. Compute \( \nabla_{\partial/\partial x^j} \partial/\partial x^k \). Have you seen that formula before?