Problem Set # 8
M392C: Riemannian Geometry

1. You may use covariant derivatives or moving frames, whichever you find more convenient.

(a) Suppose \((M, g)\) and \((M', g')\) are Riemannian manifolds. Define a Riemannian manifold \((M \times M', g \times g')\) and compute its curvature in terms of the curvatures of \((M, g)\) and \((M', g')\).

(b) Continuing, let \(f : M \to \mathbb{R}\) be a positive smooth function. Define the warped product \((M \times M', g \times f \cdot g')\) and compute its curvature. You have already seen such metrics when \(M\) and \(M'\) are each 1-dimensional.

(c) Let \((M, g)\) be a Riemannian manifold and \(\varphi : M \to \mathbb{R}\) any smooth function. Compute the curvature of \((M, e^{2\varphi} g)\). (The metric \(e^{2\varphi} g\) is conformally related to the metric \(g\).)

2. Let \(G\) be any Lie group.

(a) Use left translation to trivialize \(TG\). This defines a global parallelism on \(G\), so by differentiation a (left-invariant) covariant derivative on \(TG\). What is the curvature and torsion of this covariant derivative?

(b) Repeat with right translation replacing left translation.

(c) Recall that the Leibniz rule which defines a covariant derivative is an affine equation, so the average of two connections is a connection. Compute the curvature and torsion of the average of the connections in parts (a) and (b).

(d) You now have three connections on \(G\). Are any of these Levi-Civita connections for a metric on \(G\)?

3. Is it possible for a geodesic to intersect itself? Example or counter-proof.

4. Let \(X \subset E\) be a submanifold of a Euclidean space. The dimensions of \(X\) and \(E\) are not fixed.

(a) Use the global parallelism of \(E\) to induce a parallelism—a covariant derivative—on \(X\). So if \(\xi \in T_x X\) is a tangent vector at some point \(x \in X\) and \(\eta\) a vector field on \(X\) defined in a neighborhood of \(X\), use the natural covariant derivative on \(E\) to define the covariant derivative \(\nabla_\xi \eta\) on \(X\).

(b) Prove that \(\nabla\) preserves the induced Riemannian metric on \(X\).

(c) Consider the example of a unit 2-sphere \(X\) in a 3-dimensional Euclidean space. Let \(C\) be the circle obtained by intersecting \(X\) with a plane whose distance from the nearest parallel tangent plane is \(d < 1\). The holonomy of the parallel transport around \(C\) is rotation through some angle \(\theta\). Compute \(\theta\) as a function of \(d\). Make clear your orientations.
5. Let \( X \) be a smooth manifold and \( \pi: E \rightarrow X \) a vector bundle equipped with a covariant derivative \( \nabla \).

(a) Interpret \( \nabla \) as a linear map \( \Omega^0_X(E) \rightarrow \Omega^1_X(E) \).

(b) Use the Leibniz rule to extend to a sequence of linear maps

\[
0 \rightarrow \Omega^0_X(E) \xrightarrow{\nabla} \Omega^1_X(E) \xrightarrow{d} \Omega^2_X(E) \xrightarrow{d} \Omega^3_X(E) \rightarrow \ldots
\]

This should reduce to the de Rham complex in case \( E \) is the trivial line bundle with trivial covariant derivative.

(c) Compute \( d^2 \nabla \).

(d) Compute \( d^3 \nabla \).

6. Let \( X \) be a smooth \( n \)-manifold equipped with a connection on its frame bundle \( B(X) \rightarrow X \). (Recall that this is a principal \( GL_n\mathbb{R} \)-bundle.) Let \( \partial_k, k = 1, \ldots, n \), be the canonical horizontal vector fields on \( B(X) \). Set \( \partial = \partial_k e^k \) to be the horizontal \( (\mathbb{R}^n)^* \)-valued vector field.

(a) A differential form on \( X \) lifts to a function

\[ \omega: B(X) \rightarrow \bigwedge^*(\mathbb{R}^n)^* \]

Compute \( R^*_g \omega, g \in GL_n\mathbb{R} \).

(b) Compute \( (R_g)_*(\partial) \).

(c) Let \( \epsilon(\alpha): \bigwedge^*(\mathbb{R}^n)^* \rightarrow \bigwedge^{*+1}(\mathbb{R}^n)^* \) be exterior multiplication by \( \alpha \in (\mathbb{R}^n)^* \). Compute

\[ \epsilon(\partial)\omega = \epsilon(e^k)\partial_k \omega. \]

How does it transform under \( R_g \)? How does it compare to \( d\omega \)? In other words, compare the operators \( \epsilon(\partial) \) and \( d \). (You may want to consider functions and 1-forms first. A \( q \)-form is a sum of products of these locally. For a 1-form, lift from \( X \) to a 1-form on \( B(X) \), which we want to write as a function \( \omega_{\ell} e^{\ell} \). How can you compute \( \omega_{\ell} \) from the lifted 1-form and the vector field \( \partial_{\ell} \)?)

(d) Rewrite this exercise in terms of the covariant derivative

\[ \nabla: \Omega^q(X) \rightarrow \Gamma(X; \bigwedge^q T^*X \otimes T^*X), \]

where the codomain is the vector space of smooth sections of the indicated vector bundle over \( X \).

(e) Did this exercise shed light on why we might want a torsionfree connection?