NOTES ON LECTURE 13 (RIEMANNIAN GEOMETRY)

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Contents
I thought I should write down, at least in telegraphic form, some of the basic definitions and results about principal bundles, etc., so that you have a text to refer to. These are in part adapted from old notes... 

1. Principal bundles

Connections on principal bundles and their associated fiber bundles are a basic structure in differential geometry. Geometric structures on a manifold are encoded in a reduction of the frame bundle, and basic features of the structure are computed in terms of a connection, though these notes stop short of defining connections: see your notes from the class lectures.

(1.1) Torsors and associated spaces. Let G be a Lie group. Recall that a right G-torsor T is a manifold with a simply transitive right action of G on T. Thus for any t₀ ∈ T we have a diffeomorphism φ_{t₀}: G → T defined by φ_{t₀}(g) = t₀g. If t₁ = t₀h ∈ T is any other point, then we have the diagram of trivializations

\[
\begin{array}{ccc}
G & \xrightarrow{L_h} & G \\
\downarrow & & \downarrow \\
T & \xleftarrow{\varphi_{t_0}} & \xrightarrow{\varphi_{t_1}} T
\end{array}
\]

and the change of trivialization map φ_{t₀}⁻¹ ∘ φ_{t₁}: G → G is the left translation L_h. In other words, a right G-torsor T is identified with G (as a right G-torsor) up to a left translation. Thus any left-invariant “notions” on G are defined on a right G-torsor. For example, a left invariant vector field on G determines a vector field on T: it is the infinitesimal G-action. So every tangent space to T is canonically identified with the Lie algebra g. Dually, there is a canonical 1-form θ ∈ Ω^1_T(t) induced from the Maurer-Cartan form, and it satisfies the equation dθ + \frac{1}{2}[θ ∧ θ] = 0.

Now let F be a manifold with a left G-action. Then we can form the mixing construction or associated space

\[
F_T = T ×_G F = (T × F)/G,
\]

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where the $G$-action is the equivalence relation

\[(1.4) \quad [tg, f] = [t, gf], \quad t \in T, \ f \in F, \ g \in G.\]

Each $t_0 \in T$ gives a diffeomorphism $\psi_{t_0} : F \to F_T$ which is defined by $\psi_{t_0}(f) = [t_0, f]$. If $t_1 = t_0h \in T$, then $\psi_{t_0}^{-1} \circ \psi_{t_1} : F \to F$ is the action of $h$. Thus we have identifications of $F_T$ with $F$ up to the action of $G$.

**Example 1.5.** Let $V$ be a real $n$-dimensional vector space and $\mathcal{B}(V) = \{b : \mathbb{R}^n \to V\}$ the right $GL_n(\mathbb{R})$-torsor of bases. Then $GL_n(\mathbb{R})$ acts on $\mathbb{R}^n$ and the associated space is canonically $V$ by the map $[b, \xi] \mapsto b(\xi)$ for $\xi \in \mathbb{R}^n$. Similarly, $GL_n(\mathbb{R})$ acts on the Grassmannian $Gr_k(\mathbb{R}^n)$ of subspaces of dimension $k$ in $\mathbb{R}^n$ with associated space the Grassmannian $Gr_k(V)$. It also acts on the space of metrics on $\mathbb{R}^n$ with fixed signature, and the associated space is the corresponding space of metrics on $V$; it acts on all tensor spaces built from $\mathbb{R}^n$ with associated spaces of tensors on $V$, etc.

A positive definite metric on $V$ may be specified by a sub $O_n$-torsor $\mathcal{B}_O(V) \subset \mathcal{B}(V)$ of orthonormal frames. The space associated to the action of $O_n$ on the unit sphere $S^{n-1}(\mathbb{R}^n)$ is the unit sphere in $V$.

Heuristically, a $G$-torsor may be regarded as a space of abstract bases, or “internal states”. Working with torsors in this way is democratic: we make no choice of distinguished basis unless it is part of the geometry.

Notice in this example that $\mathbb{R}^n$ is a vector space and the $GL_n(\mathbb{R})$-action is by vector space automorphisms. Therefore, the associated space is a vector space. Quite generally, if $F$ has some structure (vector space, algebra, Lie algebra, group, etc.) preserved by the $G$-action, then $F_T$ inherits that structure.

Suppose $\rho : G \to G'$ is a homomorphism of Lie groups and $T$ a right $G$-torsor. Then $\rho$ defines a left action of $G$ on $G'$ by multiplication. This preserves the structure of $G'$ as a right $G'$-torsor, so the associated space $T \times_G G'$ is a right $G'$-torsor. This motivates the following definition.

**Definition 1.6.**

(i) Let $\rho : G \to G'$ be a homomorphism of Lie groups and $T'$ a $G'$-torsor. Then a **reduction** of $T'$ to $G$ is a pair $(T, \theta)$ consisting of a $G$-torsor $T$ and an isomorphism $\theta : T \times_G G' \to T'$ of $G'$-torsors.

(ii) If $V$ is a real vector space of dimension $n$ and $\rho : G \to GL_n(\mathbb{R})$ a homomorphism, then a **$G$-structure** on $V$ is a reduction of $\mathcal{B}(V)$ to $G$.

Equivalently, $\theta$ is a map which intertwines $\rho$. The notion of a $G$-structure on a vector space formalizes Felix Klein’s *Erlangen program*.

**Definition 1.7** **Principal bundles and associated fiber bundles.** A principal $G$-bundle over a space is a locally trivial family of $G$-torsors. Any left $G$-space $F$ then induces, by the mixing construction, a fiber bundle with structure group $G$ in the sense of Steenrod. We spell this out in the smooth context.

**Definition 1.8.** Let $M$ be a smooth manifold and $G$ a Lie group. A **principal $G$-bundle over $M$** is a smooth map $\pi : P \to M$, where the manifold $P$ is equipped with a free right $G$-action, $\pi$ is a quotient map for the $G$-action, and $\pi$ admits local sections: about each point $m \in M$ is an open neighborhood $U \subset M$ and a smooth section $s : U \to P$ of $\pi$. 
The freeness of the action means the fibers of $\pi$ are right $G$-torsors. Part of the definition is that the set of equivalence classes of the $G$-action on $P$ is the smooth manifold $M$. (For noncompact $G$ this set need not be a manifold in general, e.g. for the irrational action of $\mathbb{R}$ on the 2-torus.) A local section $s$ induces a local trivialization analogous to (1.9):

$U \times G \xrightarrow{\varphi_s} P|_U$

Here $\pi_1$ is projection onto the first factor and the diagram commutes: $\varphi_s$ is an isomorphism of $G$-torsors point by point on $M$.

**Definition 1.10.** Let $\pi: P \to M$ be a principal $G$-bundle and $F$ a smooth manifold with a left $G$-action. Then the associated fiber bundle $\pi: FP \to M$ is the quotient $FP = P \times_G F = (P \times F)/G$ defined in (1.9).

A local section of $\pi: P \to M$ induces a local trivialization $\psi_s: U \times F \to FP|_U$ by the formula $\psi_s(m, f) = [s(m), f]$. We make the important observation that a section $f$ of $\pi: FP \to M$ is equivalently a $G$-equivariant map $\tilde{f}: P \to F$; the equivariance is

$\tilde{f}(pg) = g^{-1}\tilde{f}(p), \quad p \in P, \ g \in G.$

The most important first example of a principal bundle is the bundle of frames $\pi: B(M) \to M$, which is a parametrized version of Example 1.9. It encodes the *intrinsic* geometry of a manifold $M$. The tangent bundle and all tensor bundles are associated to linear representations of $GL_n(\mathbb{R})$, assuming that $M$ has a fixed dimension $n$. For example, a vector field on $M$ is a smooth map $\xi: B(M) \to \mathbb{R}^n$ such that for all $g \in GL_n(\mathbb{R})$ we have $\xi(bg) = g^{-1}\xi(b)$. That is, we can specify a vector field as a vector-valued function on the collection of all bases.

Definition 1.9 has a straightforward parametrized generalization to the notion of reduction of structure group for principal bundles. In particular, we have the notion of a $G$-structure on a manifold as a reduction of the frame bundle. Any geometry associated to a $G$-structure may be considered *intrinsic*.

**Example 1.12.** Let $G$ be a Lie group and $H \subset G$ a closed subgroup. Then $\pi: G \to G/H$ is a principal $H$-bundle over the homogeneous space $G/H$. The tangent bundle $T(G/H)$ is associated to the linear representation of $H$ on the quotient $\mathfrak{g}/\mathfrak{h}$ of the Lie algebras. More precisely, a frame $\mathbb{R}^N \to \mathfrak{g}/\mathfrak{h}$ at the basepoint induces an $H$-structure on $G/H$. Typically $H$ is much “smaller” than $GL_{N}(\mathbb{R})$, where $N = \dim G/H$. For example, the sphere $S^{4n}$ can be presented as a homogeneous space in at least three different ways: $S^{4n} \cong O_{4n+1}/O_{4n} \cong U_{2n+1}/U_{2n} \cong Sp_{n+1}/Sp_n$. For $n = 1$ the frame bundle of $S^4$ has 16-dimensional structure group $GL_{4}(\mathbb{R})$, and the reductions to $O_4$, $U_2$, and $Sp_1$ have dimensions 6, 4, 3, respectively.
Example 1.13. Let $V$ be a real vector space, possibly infinite dimensional. (We can take $V$ complex by changing $\mathbb{R} \to \mathbb{C}$ in what follows.) For $k \leq \dim V$ define the Stiefel manifold

\[
St_k(V) = \{b : \mathbb{R}^k \to V : b \text{ is injective}\}.
\]

It has a free right action of $GL_k(\mathbb{R})$ whose quotient $Gr_k(V)$ is a manifold, the Grassmannian of $k$-dimensional subspaces of $V$. For $k = 1$ we obtain the projective space $\mathbb{P}(V)$ of lines in $V$. The tautological vector bundle of rank $k$ is associated to the standard representation of $GL_k(\mathbb{R})$ on $\mathbb{R}^k$; the fiber of the associated bundle at a $k$-plane $W \in Gr_k(V)$ is canonically identified with $W$. The tangent bundle to $Gr_k(V)$ is not associated to the Stiefel bundle $St_k(V) \to Gr_k(V)$. 