

Problem Set # 1

M392C: Topics in Geometry and Physics

Problem sets will be posted to the website, perhaps weekly. They are not to hand in. I urge you to work in groups and to come to office hours to discuss the problems and the class.

1. Suppose A, B are affine spaces. Prove that a map $f: A \rightarrow B$ is injective affine if and only if the image of every affine line in A is an affine line in B .
2. Prove that the group SO_n of orientation-preserving orthogonal transformations of \mathbb{R}^n is connected as follows. (Recall that SO_n is a group of $n \times n$ matrices.)
 - (a) Verify the result for $n = 1$ directly.
 - (b) Prove that SO_n acts transitively on the unit sphere S^{n-1} in \mathbb{R}^n . What is the stabilizer group of a point?
 - (c) Suppose that if a Lie group G acts transitively on a connected manifold X and the stabilizer group H of a point $x \in X$ is connected, then G is connected. (This is not easy from scratch.)
 - (d) Complete the proof by induction.
3. Let V be a finite dimensional real vector space.
 - (a) Prove that the space of positive definite symmetric bilinear forms is a convex cone, hence contractible. (First describe its topology. In what context is ‘convex’ defined?)
 - (b) Assume $\dim V \geq 2$. Is the space of Lorentz inner products contractible? Connected?
4. Let \mathbb{A}^2 be the standard real affine plane. Topologize the space of affine lines in \mathbb{A}^2 . Show that it is a topological manifold (and in fact can be made into a smooth manifold). What is its dimension? What can you say about its topology? Can you recognize it as homeomorphic to a familiar space? This is an example of a *moduli space*.
5. (a) Let M be a Euclidean space, that is, an affine space over a real vector space with positive definite inner product. Fix $p, q \in M$. Show that the length of the line segment joining p and q is less than or equal to the length of any piecewise linear path from p to q . Can you devise a limiting argument to show that it is smaller than the length of any smooth path from p to q ?
 - (b) Repeat for M a Minkowski spacetime and $p, q \in M$ timelike separated, i.e., $q = p + \xi$ for a timelike vector ξ . There is one very important modification: now the straight path has the *longest* proper time.

6. Let V be a Lorentzian vector space.

(a) Show that each component of the space of timelike vectors is a convex cone.

(b) Suppose that ξ, η are timelike vectors in the same component such that $\langle \eta - \xi, \xi \rangle = 0$. Let φ be the hyperbolic angle between ξ and η . Prove that

$$\tanh \varphi = \frac{|\eta - \xi|}{|\xi|}.$$

7. Let M be a Minkowski spacetime with underlying Lorentz vector space V and speed of light c . Let $\phi: V \rightarrow V^*$ be the isometry determined by the Lorentz inner product.

(a) Fix a timelike vector $\xi_0 \in V$ with $|\xi_0| = c$. Show that it determines orthogonal splittings

$$\begin{aligned} V &= \mathbb{R}\xi_0 \oplus S \\ V^* &= \mathbb{R}\theta_0 \oplus S^*, \end{aligned}$$

where $S \subset V$ has negative definite inner product. (Define θ_0 as the functional with $\theta_0(\xi_0) = 1$ and $\theta_0(S) = 0$.)

(b) Let $\zeta \in S$ and $\xi = \xi_0 + \zeta$. Write $v = |\zeta|$. Compute the θ_0 -component of $\phi(\xi)$.

(c) Recall that if $W \subset M$ is an oriented worldline of mass m_0 , then the energy-momentum at $p \in W$ is defined as $m_0\phi(\xi_p)$, where $\xi_p \in T_pM$ is the unique oriented vector with $|\xi_p| = c$. Fix affine coordinates t, x^1, \dots, x^{n-1} on M so that the metric has the form $c^2(dt)^2 - (dx^1)^2 - \dots - (dx^{n-1})^2$. Show that the energy-momentum is

$$\frac{m_0c^2}{\sqrt{1 - v^2/c^2}} dt - \sum_{i=1}^{n-1} \frac{m_0(\dot{x}^i/\dot{t})}{\sqrt{1 - v^2/c^2}} dx^i,$$

where the derivatives are with respect to any oriented parametrization of W , the square velocity in this coordinate system is $v^2 = \sum (\dot{x}^i/\dot{t})^2$, and x^i, t, v are functions on W .

8. Let t, x be coordinates on the standard Minkowski spacetime M^2 with metric $c^2dt^2 - dx^2$. Consider a Lorentz boost with hyperbolic angle φ . On the underlying vector space the associated linear transformation is given by the matrix $\begin{pmatrix} (\cosh \varphi)/c & (\sinh \varphi)/c \\ \sinh \varphi & \cosh \varphi \end{pmatrix}$. Write $v = c \tanh \varphi$. Show that we can write an associated affine transformation as

$$\begin{aligned} t' &= \frac{t + vx/c^2}{\sqrt{1 - v^2/c^2}} \\ x' &= \frac{x + vt}{\sqrt{1 - v^2/c^2}}, \end{aligned}$$

up to a translation.