Problem Set # 2

M392C: Topics in Geometry and Physics

1. Suppose
$$\begin{array}{ccc} & & M \\ & & \downarrow f \\ N & \xrightarrow{g} & P \end{array}$$
is a diagram of smooth maps with f a submersion. Prove that the fiber product $M \times_P N$ is a smooth manifold. Recall that the fiber product is defined as a submanifold of $M \times N$:
$$M \times_P N = \{(m, n) : f(m) = g(n)\}$$
and fits into the diagram
$$\begin{array}{ccc} M \times_P N & \longrightarrow & M \\ \downarrow & & \downarrow f \\ N & \xrightarrow{g} & P \end{array}$$

2. (a) Let $G \subset GL_n(\mathbb{R})$ be a subgroup and suppose M, M' are smooth manifolds equipped with a G-structure. Recall carefully the definition of a G-structure and then define what it means for a smooth map $f\colon M \to M'$ to be a map of "manifolds with G-structure". Can you extend to the case when $G \to GL_n(\mathbb{R})$ is not necessarily injective?
   (b) What is the (moduli) space of complex structures on $\mathbb{R}^{2m}$ for $m \geq 1$ an integer?
   (c) On a manifold of dimension $n = 2m$ an *almost complex structure* is a G-structure for $G = GL_m(\mathbb{C}) \subset GL_{2m}(\mathbb{R})$. Define a map of almost complex manifolds (not necessarily of the same dimension).
   (d) Construct an almost complex structure on a Lie group G of even dimension.
   (e) The only spheres to admit almost complex structures are S^2 and S^6. On the other hand, products of odd-dimensional spheres do. Construct an almost complex structure on $S^1 \times S^3$ by recognizing it as a Lie group.

3. What is the moduli "space" of Galois (or principal or regular) covering spaces of a fixed manifold M with fixed Galois group (group of deck transformations) a finite group G? In fact, it is a (Lie) *groupoid*, so present it as such. In other words, the points of this "space" possible have "internal" automorphisms, which geometricially are automorphisms of the covering space represented by the point. You may, if you like, assume that M is connected and you may find it convenient to fix a basepoint on M.

4. (a) Work on standard affine space $\mathbb{A}^n$ with coordinates $x^1, \dots, x^n$, which you may consider as local coordinates on a manifold. Let

$$\xi = \xi^i \frac{\partial}{\partial x^i} \qquad \eta = \eta^i \frac{\partial}{\partial x^i}$$

be vector fields, where ξ^i, η^i are smooth functions and the summation convention dictates a sum over the index i, which is repeated once in the numerator and denominator. Compute $[\xi, \eta]$.

(b) Prove that if ω is a q-form on a smooth manifold M, and $\xi_0, \xi_1, \dots, \xi_q$ are vector fields on M, then

$$d\omega(\xi_0, \dots, \xi_q) = \sum_{0 \le i \le q} (-1)^i \xi_i \omega(\xi_0, \dots, \not{\xi_i}, \dots, \xi_q) + \sum_{0 \le i < j \le q} (-1)^{i+j} \omega([\xi_i, \xi_j], \xi_0, \dots, \not{\xi_i}, \dots \not{\xi_j}, \dots, \xi_q).$$

You can use coordinates, for example. You may also fashion an inductive argument out of the Cartan formula $\mathcal{L}_\xi = d\iota_\xi + \iota_\xi d$. Try $q = 1$ first. In any case verify that the right hand side is *tensorial*, i.e., linear over functions.

5. Enumerate the conjugacy classes and stabilizers (centralizers) in SU_2. Repeat for O_2. What about U_2?

6. Derive the Maurer-Cartan equations

$$d\theta^i + \Theta^i_j \wedge \theta^j = 0$$
$$d\Theta^i_k + \Theta^i_j \wedge \Theta^j_k = 0$$

for the affine group. What additional equation holds for the Euclidean group?

7. A $\mathbb{Z}$-*graded Lie algebra* is a $\mathbb{Z}$-graded vector space $\mathfrak{g} = \sum_n \mathfrak{g}_n$ with a Lie bracket $[\cdot, \cdot]$ which obeys $[\mathfrak{g}_n, \mathfrak{g}_m] \subset \mathfrak{g}_{n+m}$. But the antisymmetry and Jacobi identities have some new signs according to the *Koszul sign rule* which inserts a minus sign when commuting elements of odd degree. Thus for homogeneous elements $x, y \in \mathfrak{g}$ of degrees $|x|, |y| \in \mathbb{Z}$, the (graded) skew-symmetry condition is

$$[x, y] = -(-1)^{|x||y|}[y, x].$$

(a) Write the correct Jacobi identity in the graded case.

(b) Let M be a smooth manifold and ξ a vector field on M. Construct a $\mathbb{Z}$-graded Lie algebra with $\mathfrak{g}_{-1} = \mathbb{R}\iota_\xi$, $\mathfrak{g}_0 = \mathbb{R}\mathcal{L}_\xi$, $\mathfrak{g}_1 = \mathbb{R}d$, and all other $\mathfrak{g}_n = 0$. Check all bracketing relations and the Jacobi identity carefully.