# Problem Set # 7

M392C: Topics in Geometry and Physics

1. Let $M$ be a smooth manifold of dimension $n$ and $\Delta \subset TM$ a distribution of rank $r$.
   (a) Construct a $G$-structure on $M$ with $G \subset GL_n\mathbb{R}$ the subgroup which preserves $\mathbb{R}^r \subset \mathbb{R}^n$.
   (b) Show that the $G$-structure is integrable, i.e. admits a torsionfree connection, if and only if $\Delta$ is integrable as a distribution.

2. (a) Let $S^2 \subset \mathbb{E}^3$ be the sphere of unit radius with the induced metric. Show that the bundle $\mathcal{B}_O(S^2)$ of orthonormal frames is a torsor for the orthogonal group $O_3$.
   (b) Let $B^2 \subset \mathbb{A}^2$ be the open unit disk in the affine plane with Riemannian metric

$$ds^2 = \frac{(dx^1)^2 + (dx^2)^2}{1 - (x^1)^2 - (x^2)^2}.$$

   Show that its orthornomal frame bundle is a torsor for the orthogonal group $O_{1,2}$ of isometries of a metric of signature $(1, 2)$.

3. Recall the structure equations

$$d\theta^i + \Theta^i_j \wedge \theta^j = \tau^i$$
$$d\Theta^i_j + \Theta^i_k \wedge \Theta^k_j = \Omega^i_j$$

   for a connection on the frame bundle of a smooth manifold. Differentiate these equations to derive the *Bianchi identities* for the *covariant* derivatives of $\tau$ and $\Omega$. You may work with the equations in matrix form. Recall the covariant derivatives are

$$d_\Theta \tau = d\tau + \Theta \wedge \tau$$
$$d_\Theta \Omega = d\Omega + [\Theta \wedge \Omega].$$

4. Let $M$ be a Riemannian manifold and $\theta^1, \dots, \theta^n$ a local orthonormal moving frame. Suppose given 2-forms

$$\tau^i = \frac{1}{2} T^i_{jk} \theta^j \wedge \theta^k$$

   with $T^i_{jk}$ skew in $j, k$. Solve for the unique orthogonal connection with torsion $\tau$. (An orthogonal connection is a set of 1-forms $\Theta^i_j$ with $\Theta^j_i = -\Theta^i_j$. You will want to write $\Theta^i_j = \Gamma^i_{jk}\theta^k$ in terms of the local basis of 1-forms.) Repeat for a local moving frame $\partial/\partial x^1, \dots, \partial/\partial x^n$ associated to a local coordinate system $x^1, \dots, x^n$.

5. Find an example of each of the following.

   (a) A manifold $M$ with linear connection so that each geodesic (a parametrized curve) is defined for all time in both directions.

   (b) A manifold $M$ with linear connection so that each geodesic (a parametrized curve) is not defined for all time in both directions.

   (c) A manifold $M$ with linear connection such that the curvature is constant and nonzero. (For this purpose view the curvature as a set of functions $R^i_{jk\ell}$ on the frame bundle; these functions should be constant.)

   (d) A manifold $M$ with linear connection such that the curvature is not constant.

   (e) A manifold $M$ with linear connection and a loop $S^1 \to M$ such that there is no parallel vector field around the loop (i.e., no parallel section of the pullback of $TM$).

6. Let $G$ be a Lie group.

   (a) Show that left translation defines a linear connection which is flat (zero curvature). Compute its torsion.

   (b) Show that right translation defines a linear connection which is flat (zero curvature). Compute its torsion.

   (c) Suppose $G$ admits a bi-invariant Riemannian metric. (This is always the case if $G$ is compact.) Prove that it is the midpoint on the affine line determined by the connections in (a) and (b). (Recall that the space of linear connections is an affine space.) What is its curvature? Torsion is an affine linear function and curvature an affine quadratic function of the connection. Check that for these three points on this distinguished affine line of connections.