Remarks on Fully Extended 3-Dimensional Topological Field Theories

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Work in progress with Constantin Teleman
Manifolds and Algebra: Abelian Groups

Pontrjagin and Thom introduced abelian groups $\Omega_k$ of manifolds called bordism groups—equivalence classes of closed $k$-manifolds:

The abelian group operation is disjoint union. (Cartesian product defines a ring structure on $\oplus_k \Omega_k$)

Thom showed how to compute $\Omega_k$ via homotopy theory: $\Omega_k = \pi_k MO$. Different answers for different flavors of manifolds: oriented, spin, almost complex, framed, ... 

Many applications in homotopy theory: (i) framed bordism groups are stable homotopy groups of spheres (Pontrjagin-Thom construction); (ii) complex cobordism is universal among certain cohomology theories (Quillen)
A more elaborate algebraic structure is obtained if we (i) do not identify bordant manifolds and (ii) remember the bordisms. So fix $n$ and introduce a bordism category $\text{Bo}_n$ whose objects are closed $(n - 1)$-manifolds and morphisms are compact $n$-manifolds $X : Y_0 \to Y_1$.

Identify diffeomorphic bordisms. Composition is gluing of manifolds:

Disjoint union gives a symmetric monoidal structure on $\text{Bo}_n$.

Recover the abelian group $\Omega_{n-1}$ by declaring all morphisms to be invertible=iso morphisms. New information from non-invertibility.
**Topological Quantum Field Theory**

\( \textbf{Vect}_\mathbb{C} = \) symmetric monoidal category (\( \otimes \)) of complex vector spaces.

**Definition:** An \( n \)-dimensional TQFT is a homomorphism

\[
F : \text{Bo}_n \longrightarrow \textbf{Vect}_\mathbb{C}
\]

This slick definition encodes an algebraic understanding of field theory descended from Witten, Quillen, Segal, Atiyah, . . . . Segal defines conformal and more general QFTs via geometric bordism categories.

**locality** (compositions)

**multiplicativity** (monoidal structure)

\( E_n \)-algebra structure on \( S^{n-1} \in \text{Bo}_n \) via the generalized “pair of pants”:

\[
D^n \backslash (D^n \sqcup D^n) : S^{n-1} \sqcup S^{n-1} \longrightarrow S^{n-1}
\]

Therefore, \( F(S^{n-1}) \in \textbf{Vect}_\mathbb{C} \) is also an \( E_n \)-algebra (OPE)
Manifolds and Algebra: Topological Categories

A more refined version of $B_n$ is a topological category with a space of $n$-dimensional bordisms between fixed $(n-1)$-manifolds.

Algebraic topology provides a construction which inverts all morphisms: topological category $\rightarrow$ topological space. With abelian group structure: symmetric monoidal topological category $\rightarrow$ spectrum.

Galatius-Madsen-Tillmann-Weiss (2006) identify the spectrum $|B_n| = MT$. For framed manifolds it is the sphere spectrum.

Remark: Topological spaces give rise to (higher) categories in which all morphisms are invertible: the fundamental groupoid

Definition (F.-Moore): A field theory $\alpha: B_n \rightarrow \text{Vect}_\mathbb{C}$ is invertible if $\alpha(Y^{n-1}) \in \text{Vect}_\mathbb{C}$ is a line and $\alpha(X^n)$ is an isomorphism between lines for all $Y, X$.

$\alpha$ factors through $MT$ spectrum $\Rightarrow$ homotopy theory techniques
Extended Field Theories

The notion of extended QFT was explored in various guises in the early ’90s by several mathematicians and has great current interest:

- 2-dimensional theories often include categories attached to a point: D-branes, Fukaya category, ...
- 4-dimensional supersymmetric gauge theories have categories of line operators. Also, the category attached to a surface plays a key role in the geometric Langlands story.
- Chern-Simons (1-2-3) theory $F$ has a linear category $F(S^1)$. For gauge group $G$ two descriptions of $F(S^1)$: positive energy representations of $LG$, or representations of a quantum group.

A fully extended theory (down to 0-manifolds) is completely local $\Rightarrow$ powerful computational techniques, simpler classification

**Longstanding Question:** Can 1-2-3 Chern-Simons theory be extended to a 0-1-2-3 theory? If so, what is attached to a point?

Manifolds and Algebra: $(\infty, n)$-Categories

A new algebraic gadget: the bordism $(\infty, n)$-category $\text{Bord}_n$.

$\text{Bo}_n$: $(n - 1)$-manifolds and $n$-manifolds with boundary
$\text{Bord}_n$: 0-, 1-, ..., $n$-manifolds with corners

Objects are compact 0-manifolds, 1-morphisms are compact 1-manifolds with boundary, 2-morphisms are compact 2-manifolds with corners, ...

![Diagram](attachment:image.png)

**Definition:** An extended $n$-dimensional TQFT is a homomorphism

$$F: \text{Bord}_n \longrightarrow \mathcal{C}$$

for some $(\infty, n)$-category $\mathcal{C}$.

For example, if $n = 3$ then typically $F(S^1)$ is a $\mathcal{C}$-linear category, also an $E_2$-algebra. $E_2(\text{Cat}_\mathcal{C}) = \beta \otimes \text{Cat}_\mathcal{C}$ are braided tensor categories.
The Cobordism Hypothesis

A powerful theorem in topological field theory, conjectured by Baez-Dolan then elaborated and proved by Lurie (w/Hopkins for \( n = 2 \)), asserts that an extended TQFT is determined by its value on a point.

**Theorem:** For framed manifolds the map

\[ \text{Hom}(\text{Bord}_n, \mathcal{C}) \longrightarrow \mathcal{C} \]

\[ F \longmapsto F(\text{pt}) \]

is an isomorphism onto the fully dualizable objects in \( \mathcal{C} \).

**Remark:** This is really a theorem about framed \( \text{Bord}_n \), asserting that it is freely generated by a single generator.

The proof, only sketched heretofore, has at its heart a contractibility theorem in *Morse theory* (Igusa, Galatius). There are variations for other bordism categories of manifolds, also manifolds with singularities.

Full dualizable is a finiteness condition. For example, in a TQFT the vector spaces attached to closed \((n - 1)\)-manifolds are finite dimensional. In an extended theory \( F(\text{pt}) \) satisfies analogous finiteness conditions.
Spheres and Invertibility

**Theorem (F.-Teleman):** Let $\alpha : \text{Bord}_n \to \mathcal{C}$ be an extended TQFT such that $\alpha(S^k)$ is invertible. Then if $n \geq 2k$ the field theory $\alpha$ is invertible.

Thus $\alpha(X)$ is invertible for all manifolds $X$. This means $\alpha$ factors through the Madsen-Tillmann spectrum constructed from $\text{Bord}_n$, so is amenable to homotopy theory techniques.

**Remark 1:** We have only checked the details carefully for oriented manifolds; it is probably true for stably framed manifolds as well.

**Remark 2:** Again this is a theorem about $\text{Bord}_n$, asserting that if we localize by inverting $S^k$, then every manifold is inverted.

**Remark 3:** As I explain later we apply this to $n = 4$, $k = 2$, and $\mathcal{C} = \beta \otimes \text{Cat}_C$ the symmetric monoidal 4-category of braided tensor categories. Then $\alpha$ is the anomaly theory for Chern-Simons, and we construct Chern-Simons as a 0-1-2-3 anomalous theory.
Proof Sketch

First, by the cobordism hypothesis (easy part) it suffices to prove that $\alpha(\text{pt}_+)$ is invertible; ‘+’ denotes the orientation. We omit ‘$\alpha$’ and simply say ‘$\text{pt}_+$ is invertible’.

We prove the 0-manifolds $\text{pt}_+$ and $\text{pt}_-$ are inverse:

\[ S^0 = \text{pt}_+ \amalg \text{pt}_- = \text{pt}_+ \otimes \text{pt}_- \cong \emptyset^0 = 1 \]

with inverse isomorphisms given by

\[ f = D^1 : 1 \longrightarrow S^0 \]
\[ f^\vee = D^1 : S^0 \longrightarrow 1 \]

We arrive at a statement about 1-manifolds: the compositions

\[ f^\vee \circ f = S^1 : 1 \longrightarrow 1 \]
\[ f \circ f^\vee : S^0 \longrightarrow S^0 \]

are the identity.
Let's now consider \( n = 2 \) where we assume that \( S^1 \) is invertible. We apply an easy algebraic lemma which asserts that invertible objects are dualizable and the dualization data is invertible. For \( S^1 \) these data are dual cylinders, and so the composition \( S^1 \times S^1 \) is also invertible.

**Lemma:** Suppose \( \mathcal{D} \) is a symmetric monoidal category, \( x \in \mathcal{D} \) is invertible, and \( g: 1 \to x \) and \( h: x \to 1 \) satisfy \( h \circ g = \text{id}_1 \). Then \( g \circ h = \text{id}_x \) and so each of \( g, h \) is an isomorphism.

**Proof sketch:** \( x^{-1} \) is a dual of \( x \), \( g^\vee = x^{-1}g: x^{-1} \to 1 \), \( h^\vee = x^{-1}h: 1 \to x^{-1} \), so the lemma follows from \( (h \circ g)^\vee = \text{id}_1 \).

Apply the lemma to the 2-morphisms

\[
g = D^2: 1 \longrightarrow S^1
\]
\[
h = S^1 \times S^1 \backslash D^2: S^1 \longrightarrow 1
\]

Conclude that \( S^1 \cong 1 \) and \( S^2 = g^\vee \circ g \) is invertible. Also, \( g \circ g^\vee = \text{id}_{S^1} \otimes S^2 \), a simple surgery.
Recall that we must prove that the compositions

\[ f^\vee \circ f = S^1 : 1 \to 1 \]

\[ f \circ f^\vee : S^0 \to S^0 \]

are the identity. We just did the first.

For the second the identity is and we will show that the saddle

\[ \sigma : f \circ f^\vee \to \text{id}_{S^0} \]

is an isomorphism with inverse \( \sigma^\vee \otimes S^2 \).

The saddle \( \sigma \) is diffeomorphic to \( D^1 \times D^1 \), which is a manifold with corners. Its dual \( \sigma^\vee \) is the time-reversed bordism.
Inside each composition $\sigma^\vee \circ \sigma$ and $\sigma \circ \sigma^\vee$ we find a cylinder $\text{id}_{S^1} = D^1 \times S^1$, which is $(S^2)^{-1} \otimes g \circ g^\vee = (S^2)^{-1} \otimes (S^0 \times D^2)$ by a previous argument. Making the replacement we get the desired isomorphisms to identity maps.

This completes the proof of the theorem in $n = 2$ dimensions.

In higher dimensions we see a kind of Poincaré duality phenomenon: we prove invertibility by assuming it in the middle dimension. A new ingredient—a dimensional reduction argument—also appears.
Application to Modular Tensor Categories

Let $F$ denote the usual quantum Chern-Simons 1-2-3 theory for some gauge group $G$. It was introduced by Witten and constructed by Reshetikhin-Turaev from quantum group data. The latter construction works for any modular tensor category $A$, a braided tensor category which satisfies finiteness conditions (semisimple with finitely many simples, duality, etc.) and a nondegeneracy condition (the $S$ matrix is invertible). Then $F$ is a 1-2-3 theory with $F(S^1) = A$.

Let $A$ be a braided tensor category with braiding $\beta(x, y): x \otimes y \to y \otimes x$.

**Theorem:** The nondegeneracy condition on $A$ is equivalent to

$$\{ x \in A : \beta(y, x) \circ \beta(x, y) = \text{id}_{x \otimes y} \text{ for all } y \in A \} = \{ \text{multiples of } 1 \in A \}.$$ 

This is proved by Müger and others.

Recall that $\beta \otimes \text{Cat}_\mathcal{C} = E_2(\text{Cat}_\mathcal{C})$. 

Braided tensor categories form the objects of a 4-category!

<table>
<thead>
<tr>
<th>object</th>
<th>category number</th>
</tr>
</thead>
<tbody>
<tr>
<td>element of $\mathbb{C}$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\mathbb{C}$-vector space</td>
<td>$0$</td>
</tr>
<tr>
<td>$\text{Vect}_\mathbb{C}$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\text{Cat}_\mathbb{C}$</td>
<td>$2$</td>
</tr>
<tr>
<td>$\otimes\text{Cat}<em>\mathbb{C} = E_1(\text{Cat}</em>\mathbb{C})$</td>
<td>$3$</td>
</tr>
<tr>
<td>$\beta \otimes \text{Cat}<em>\mathbb{C} = E_2(\text{Cat}</em>\mathbb{C})$</td>
<td>$4$</td>
</tr>
</tbody>
</table>

**Morita:** Morphisms of tensor categories are bimodules and morphisms of braided tensor categories are tensor categories which are bimodules.

So, given sufficient finiteness, a braided tensor category determines (using the cobordism hypothesis) an extended 4-dimensional TQFT

$$\alpha: \text{Bord}_4 \rightarrow \beta \otimes \text{Cat}_\mathbb{C}$$
In the theory $\alpha$ we compute

$$\alpha(S^2) = \{ x \in A : \beta(y, x) \circ \beta(x, y) = \text{id}_{x \otimes y} \text{ for all } y \in A \} \in \text{Cat}_C$$

Recall that for a modular tensor category this “higher center” of $A$ is the tensor unit $1 = \text{Vect}_C$, which in particular is invertible.

Thus, modulo careful verification of finiteness conditions, we have

**Corollary:** A modular tensor category $A \in \beta \otimes \text{Cat}_C$ is invertible, so determines an invertible field theory $\alpha : \text{Bord}_4 \to \beta \otimes \text{Cat}_C$.

**Remark:** This is a theorem in algebra, proved using the universal “algebra”, rather $(\infty, n)$-category, of manifolds with corners.

We believe that this is the anomaly theory for a 0-1-2-3 extension of the 1-2-3 theory $F$ with $F(S^1) = A$. In the remainder of the lecture I will explain this idea.
Anomalous Field Theories

The top-level values of an $n$-dimensional field theory $F: \text{Bo}_n \to \text{Vect}_\mathbb{C}$ are complex numbers $F(X^n) \in \mathbb{C}$, the partition function of a closed $n$-manifold. In an anomalous field theory $f$ there is a complex line $L_X$ associated to $X$ and the partition function $f(X) \in L_X$ lies in that line.

The lines $L_X$ obey locality and multiplicativity laws, so typically belong to an $(n+1)$-dimensional invertible field theory $\alpha: \text{Bo}_{n+1} \to \text{Vect}_\mathbb{C}$.

$f$ is an $n$-dimensional theory with values in the $(n+1)$-dimensional theory $\alpha$. We write $f: 1 \to \alpha$ in the sense that $f(X): 1 \to \alpha(X)$ for all $X$. (1 is the trivial theory.) If $\alpha$ is invertible we say $f$ is anomalous with anomaly $\alpha$. The same ideas apply in extended field theories.

Remark: The notion of $\alpha$-valued field theory makes sense even if $\alpha$ is not invertible and also for non-topological field theories. Examples:

(i) $n = 2$ chiral WZW valued in topological Chern-Simons, (ii) $n = 6$ $(0,2)$-(super)conformal field theory valued in a 7-dimensional theory.

Remark: This is a specialization of the notion of a domain wall.
Fully Extended Chern-Simons

Recall that a modular tensor category $A$ determines an invertible extended field theory $\alpha: \text{Bord}_4 \to \beta \otimes \text{Cat}_C$ with values in the 4-category of braided tensor categories, or equivalently $E_2$-algebras in $\text{Cat}_C$.

An ordinary algebra $A$ is in a natural way a left $A$-module. This holds for $E_2$-algebras, and in that context the module defines a morphism $A: 1 \to A$ in the 4-category $\beta \otimes \text{Cat}_C$.

Let $A$ be a modular tensor category. Modulo careful verification of finiteness conditions, a version of the cobordism hypothesis constructs from the module $A$ a 0-1-2-3-dimensional anomalous field theory $f: 1 \to \alpha$ with anomaly $\alpha$.

**Claim:** On 1-, 2-, and 3-dimensional manifolds we can trivialize the anomaly $\alpha$ and so identify $f$ with the Reshetikhin-Turaev 1-2-3 theory $F$ associated to the modular tensor category $A$.

For example, the composition $1 \xrightarrow{f(S^1)} \alpha(S^1) \xrightarrow{\alpha(D^2)} 1$ is $F(S^1) = A$, where the bordism $D^2: S^1 \to 1$ is used to trivialize the anomaly on $S^1$. 