

Math 392C: Riemannian Geometry

Fall Semester 2007

Unique Number 60535

Introduction

This course is an introduction to Riemannian geometry. It is intended for students already familiar with topological and differentiable manifolds. The objects of Riemannian geometry are smooth manifolds equipped with extra structures that provide geometric information. In particular, we will equip a manifold with a *Riemannian metric* that allows measurement of quantities such as distance and angle, and an *affine connection* that allows comparison and differentiation of tensor (e.g. vector) fields. These structures can be constrained by the topology of the manifold, hence may not be prescribed arbitrarily. Such constraints often manifest themselves through the key idea of *curvature*, which measures, in a sense, how the geometry of a given smooth manifold differs from that of Euclidean space.

Text

Our text will be:

John M. Lee. *Riemannian Manifolds: An Introduction to Curvature*. Springer-Verlag, New York, 1997.

Workload

There will be eight homework assignments during the semester, each comprising a small number of short problems. The main purpose of these exercises will be to enhance understanding of topics discussed in class. They will be graded on a credit/no-credit basis: I'll give credit for a reasonable attempt at completion.

During the semester, I will also distribute a list of projects. Each student desiring an A will be asked to prepare one 25-minute presentation on one of these assigned projects.

Outline

We will cover the following topics, investigating the most important ones in detail and at least introducing the others.

Chapter 0 Differentiable manifolds

- Tensor fields, bundles
- Differential forms

Chapter 1 Riemannian metrics

- Induced metrics on tensor bundles
- Metric inverse

Chapter 2 Connections and covariant differentiation

- Differentiation of tensor fields
- Higher derivatives
- Lie and exterior derivatives

Chapter 3 Geodesics

- Variational approach; Euler–Lagrange equation
- Injectivity radius
- Completeness, Hopf–Rinow

Chapter 4 Curvature

- Bianchi identities
- Ricci identities
- Bochner formulas
- Decompositions
- Examples

Chapter 4-1/2 Integration

- Applications of Bochner formulas
- Eigenvalues, eigenfunctions, eigentensors
- Hodge theory

Chapter 5 Jacobi fields

- Riccati equations
- Differentiation of distance
- Laplacian comparison

Chapter 6 Isometric immersions

- Riemannian submersions
- Variation of area, volume

Chapter 7 Complete manifolds

- cut loci
- injectivity radius estimates

Chapter 8 Spaces of constant curvature

- Homogeneous spaces
- Symmetric spaces

Chapter 9 Variations of length, energy

- Applications to comparison geometry
- Stability of (long) geodesics

Chapter 10 Rauch comparison theorem

- Toponogov comparison theorem