FREE GROUPS IN LATTICES (EXTENDED ABSTRACT)

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The surface subgroup conjecture posits that if Γ is a cocompact Kleinian group then Γ contains a surface subgroup; that is, a subgroup isomorphic to the fundamental group of a closed surface of genus at least 2. This is implied by the Virtual Haken conjecture (due to Thurston) and is of central importance to the classification of hyperbolic 3-manifolds.

Here is a more general problem. Let Γ be a discrete cocompact subgroup of G, a locally compact unimodular group. Prove the existence of subgroups of Γ satisfying prescribed properties. These properties may be of a group theoretic nature or an asymptotic-geometric nature.

The point behind this problem is that it might be possible to use knowledge of G to infer the existence of certain kinds of subgroups of Γ. This talk (and the paper [Bo07]) explores the following strategy for attacking this problem: given a subgroup $F < G$, we attempt to "perturb" F slightly so that some finite-index subgroup of the perturbed group lies inside Γ. For this to be successful, we would like the perturbed subgroup to have isomorphism type close to that of F and to have asymptotic geometric properties close to that of F.

To be precise, let F be an abstract group and $\phi : F \to G$ a homomorphism. Let $S \subset F$ be a finite symmetric generating set. Let d be a left-invariant proper metric on G. For $\epsilon > 0$, we will say that a map $\phi_\epsilon : F \to G$ is an **ϵ-perturbation** of F if

$$d(\phi_\epsilon(g)^{-1}\phi_\epsilon(gs), \phi(s)) = d(\phi_\epsilon(gs), \phi_\epsilon(g)s) \le \epsilon$$

for all $g \in F$ and $s \in S$. ϕ_ϵ need not be a homomorphism. Indeed, we do not even require that it maps the identity element to the identity element.

For example, if $G = \mathbb{R}$, $F = \mathbb{Z}$ and $\phi : \mathbb{Z} \to \mathbb{R}$ is the homomorphism $\phi(n) = n\tau$ for some number $\tau > 0$ then $\phi_\epsilon : \mathbb{Z} \to \mathbb{R}$ need only satisfy $|\phi_\epsilon(n+1) - \phi_\epsilon(n) - \tau| \le \epsilon$ for all $n \in \mathbb{Z}$.

We say that $\phi_\epsilon : F \to G$ is *virtually a homomorphism into* Γ if there exists a finite index subgroup $F' < F$ such that

$$\phi_\epsilon(f_1 f_2) = \phi_\epsilon(f_1)\phi_\epsilon(f_2) \qquad \forall f_1 \in F',\ f_2 \in F$$

and $\phi_\epsilon(F') < \Gamma$.

Theorem 0.1. *(Main theorem) Suppose G, d, F, S and ϕ are as above. Suppose F is free, finitely generated and S is a symmetric free generating set for F. Then for every $\epsilon > 0$ there exists an ϵ-perturbation ϕ_ϵ of ϕ that is virtually a homomorphism into Γ.*

I do not know if the theorem remains true if F is required to be a surface group instead.

0.1. **Asymptotic Geometry.** Claim: if ϵ is small and F is "nice" then an ϵ-perturbation of F does not change F very much. To be specific, assume that $G = SO(n,1)$ and F is a convex cocompact free subgroup of G.

Let $\mathbb{H}^n$ denote hyperbolic n-space. So G is the group of orientation preserving isometries of $\mathbb{H}^n$. Let S^{n-1}_∞ denote the boundary at infinity. For any $p \in \mathbb{H}^n$, let $Fp = \{fp : f \in F\}$.

The closure of Fp in $\mathbb{H}^n \cup S^{n-1}_\infty$ is denoted by $\overline{Fp}$. Its intersection with S^{n-1}_∞ is the *limit set of* F, denoted $L(F)$. It does not depend on p. Let $D(F)$ denote its Hausdorff dimension.

Similarly, if ϕ_ϵ is an ϵ-perturbation of the inclusion map $F \to G$, let $\phi_\epsilon(F)p = \{\phi_\epsilon(f)p : f \in F\}$ and let $L(\phi_\epsilon(F))$ be the intersection of S^{n-1}_∞ with the closure $\overline{\phi_\epsilon(F)p}$. Let $D(\phi_\epsilon(F))$ denote its Hausdorff dimension.

Theorem 0.2. *Let $F < SO(n,1)$ be a free convex cocompact subgroup. For every $\epsilon > 0$ let ϕ_ϵ be an ϵ-pertubation of the inclusion map $F \to G$. Then,*

1. *for all $\epsilon > 0$ sufficiently small, ϕ_ϵ is 1-1;*
2. *if $\phi_\epsilon(id) = id$ (where id denotes the identity element) then $L(\phi_\epsilon(F))$ converges to $L(F)$ in the Hausdorff topology as $\epsilon \to 0$.*

Moreover, if ϕ_ϵ is a virtual homomorphism then $D(\phi_\epsilon(F)) \to D(F)$ as $\epsilon \to 0$.

0.2. **Applications.** If H is any subgroup of $G = SO(n,1)$, let $D_{free}(H)$ denote the set of all numbers of the form $D(F)$ where F is a free, convex cocompact subgroup of H.

Theorem 0.3. *If Γ is a lattice in $G = SO(n,1)$ then $\overline{D_{free}(\Gamma)} = \overline{D_{free}(G)}$.*

Remark 1. It is easy and well-known that $D_{free}(SO(2,1)) = (0,1)$. From work of Thurston and others on geometrically infinite free groups it can be proven that $D_{free}(SO(3,1)) = (0,2)$. It is not known whether these results extend to $SO(n,1)$ for $n \geq 4$.

The Cheeger constant of a closed Riemannian manifold M is defined by

$$h(M) := \inf_S \frac{area(S)}{\min(vol(X_1), vol(X_2))}.$$

where S varies over all codimension 1 submanifolds that divide M into two pieces, X_1 and X_2. If M is noncompact then

$$h(M) := \inf_X \frac{area(\partial X)}{vol(X)}.$$

where X varies over all compact codimension 0 submanifolds of M.

Theorem 0.3 was recently employed by Lackenby, Long and Reid [LLR08] to obtain the next two theorems.

Theorem 0.4. *If M is a closed hyperbolic 3-manifold then there exists a sequence of infinite-sheeted coverings M_i of M such that $h(M_i) \to 0$.*

Theorem 0.5. *If M is a closed hyperbolic 3-manifold and $\pi_1(M)$ is LERF then there exists a sequence of finite coverings M_i of M such that $h(M_i) \to 0$. I.e., $\pi_1(M)$ does not have property τ. I.e., the Lubotzky-Sarnak conjecture holds for $\pi_1(M)$.*

Theorem 0.3 was recently used by Lackenby [La08] to prove:

Theorem 0.6 (Lackenby)**.** *If $\Gamma < SO(3,1)$ is discrete, finitely generated and contains a noncyclic finite subgroup then either Γ is finite, Γ is virtually free or Γ contains a surface subgroup.*

References

[Bo07] L. Bowen. *Free Groups in Lattices*. arXiv:0802.0185, submitted to Geometry and Topology.
[La08] M. Lackenby. *Surface subgroups of Kleinian groups with torsion*. arXiv:0804.1309.
[LLR08] M. Lackenby, D. Long and A. Reid. *LERF and the Lubotzky-Sarnak conjecture*. Geom. Topol. 12 (2008), 2047–2056.