

Differentiable structures on metric measure spaces

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A classical theorem of Rademacher asserts that a real valued Lipschitz function on \mathbf{R}^n is differentiable almost everywhere with respect to Lebesgue measure. The statement incorporates three notions:

- *Lipschitz*, which is defined for any metric space, (X, d^X) .
- *Almost everywhere*, which is defined for any measure space, (X, μ) .
- *Differentiability*, which classically was defined only for real-valued functions on spaces which are locally bi-Lipschitz to \mathbf{R}^n .

In the late 1990's we gave a notion of (first order) *differentiable structure* on any metric measure space (X, d^X, μ) . When it exists, this structure is unique up to bi-Lipschitz equivalence. We showed that if the measure μ is doubling and a Poincaré inequality holds in the upper gradient sense, then such a differentiable structure exists. The Poincaré inequality is equivalent to the existence of “sufficiently many” curves of finite length (in a sense which can be made precise). We call such spaces *PI spaces*.

Examples of spaces of PI spaces include fractals such as Carnot groups with Carnot–Carathéodory metrics and Laakso spaces. Although these spaces are not infinitesimally bi-Lipschitz to \mathbf{R}^n , we showed they do possess a finite dimensional (measurable) cotangent bundle which is the receiving space for differentials of Lipschitz functions. Classical fractals such as the Sierpinski gasket are not PI spaces because they do not contain sufficiently many curves of finite length (although every pair of points is joined by a minimal geodesic).

The theory has had applications which are not internal to the theory itself. Here we emphasize the connection with bi-Lipschitz nonembedding theorems. We showed (roughly speaking) that a PI space which is bi-Lipschitz to a subset of some \mathbf{R}^N must look quite “tame”. In particular, the above mentioned fractal spaces do not bi-Lipschitz embed in \mathbf{R}^N for any $N < \infty$.

The above suggests the problem of giving conditions under which a given PI space does or does not bi-Lipschitz embed in particular infinite dimensional Banach spaces. This question, which is of considerable interest in theoretical computer science, has been addressed in joint work with Bruce Kleiner.

It was clear initially, that if one could extend the differentiation theory to Lipschitz functions, $f : X \rightarrow V$, for some Banach space V , then the bi-Lipschitz nonembedding

theorem (and its proof) would carry over as well. This was done several years ago for separable dual space targets. Recently, we proved the optimal possible result. Namely, let V have the property that all Lipschitz functions $f : \mathbf{R} \rightarrow V$ are differentiable. (In classical terminology, V is said to possess the *Radon–Nikodym Property*.) Then all Lipschitz functions $f : X \rightarrow V$ are differentiable for any PI space X . The proof of this theorem led to the uncovering of additional ways in which PI spaces resemble \mathbf{R}^n at the infinitesimal level (although, once again, such spaces can have fractional Hausdorff dimension).

There is a simple classical example of a Lipschitz map $\mathbf{R} \rightarrow L^1$ which is *not* differentiable anywhere, i.e. L^1 does not have the Radon–Nikodym Property. Moreover, in the context of theoretical computer science, questions of bi-Lipschitz embedding in L^1 play a distinguished and important role. For the domains \mathbf{R}^n and the Heisenberg group equipped with its Carnot–Carathéodory metric, Kleiner and I showed (their possible nondifferentiability notwithstanding) that Lipschitz functions to L^1 *are* differentiable in a weakened sense. For the Heisenberg group this *does* preclude the existence of a bi-Lipschitz embedding in L^1 . Our result is based on a connection between Lipschitz maps $f : X \rightarrow L^1$ and subsets of X with *finite perimeter*. (These are classical objects of study in geometric measure theory). The notion of differentiability and the connection with sets of finite perimeter are new even for the domain \mathbf{R}^n .

The above work gave a natural counter example to the Goemans–Linial conjecture of theoretical computer science. (The first such was given slightly earlier by Khot–Vishnoi.). In joint work with Kleiner and Assaf Naor, we show that this counter example is qualitatively stronger than the previous one and in fact, is qualitatively close to being best possible.