The space of negatively-curved metrics on a closed manifold

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Abstract: The talk was a report on joint work with Pedro Ontaneda.

Let M be a closed (connected) smooth manifold of dimension n. Let $\mathrm{Met}(M,-)$ denote the space of all Riemannian metrics on M whose sectional curvatures are all negative. And let $C(p)$ denote the infinite abelian group, which is the countable direct sum of cyclic groups of prime order p.

1 Theorem *Assume $n > 9$ and $\mathrm{Met}(M,-)$ is nonempty. Then:*

1. $\mathrm{Met}(M,-)$ *has infinitely many path components K.*
2. *For each component K and each prime $p > 2$, $C(p)$ is a subgroup of the $2p-4$ homotopy group of K provided that $n > 4p+2$.*
3. *Also, $C(2)$ is a subgroup of the fundamental group of K provided $n > 13$.*

In particular, Part 1 of Theorem 1 solves "Question 7.1" on the 1984 list compiled by Burns and Katok about non-positively curved manifolds.

The Teichmüller and Moduli spaces of M are quotient spaces of $\mathrm{Met}(M,-)$ obtained by identifying isometric metrics in the case of the Moduli space and identifying metrics isometric via isometries homotopic to the identity map in the case of the Teichmüller space. Results about the homotopy groups of these spaces were also mentioned together with some applications to the study of smooth bundles with negatively curved fibers diffeomorphic to M and having a fixed base space B. Two such bundles are said to be equivalent if there is such a bundle with base $B \times [0,1]$ whose restriction to $B \times 0$ and $B \times 1$ are fiberwise isometric to the respective given bundles. It is natural is ask whether the "forget structure" map F to the underlying smooth M-bundles is (1) non-trivial, (2) onto, or (3) one-to-one. In the case the base space B is a sphere, Theorem 1 together with our results on the Teichmüller space of M show that F is frequently non-trival, but usually not onto, and frequently not one-to-one. And our results lead us to conjecture that such a bundle must be topologically trivial when B is simply connected.