

Harmonic maps and the stabilization of Heegaard splittings

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Abstract: Area minimizing surfaces have proved to be a powerful tool in the study of 3-dimensional manifolds. Harmonic maps of surfaces to 3-manifolds have not been as widely applied, due to several limitations. A homotopy class of surfaces gives rise to a large space of harmonic maps, one for each conformal structure on its domain. These maps do not minimize their self-intersections, and may fail to be immersed. Harmonic maps are somewhat better behaved, however, when the 3-manifold is negatively curved. In this setting smooth families of surfaces give rise to smooth families of harmonic maps, a property that cannot be expected from minimal surfaces, and that can be used to explore topological properties of the manifold. In this paper we study Heegaard splittings of 3-manifolds via families of harmonic surfaces.

A *genus g Heegaard splitting* of a 3-manifold M is a decomposition of M into two genus g handlebodies with a common boundary. It is described by an ordered triple (H_1, H_2, M) where each of H_1, H_2 is a handlebody in M and the two handlebodies intersect along their common boundary S , called a *Heegaard surface*. Two Heegaard splittings (H_1, H_2, M) and (H'_1, H'_2, M) are equivalent if an ambient isotopy of M carries (H_1, H_2) to (H'_1, H'_2) . Every 3-manifold has a Heegaard splitting and Heegaard splittings form one of the basic structures used to analyze and understand 3-manifolds.

Corresponding to the Heegaard splitting is a family of surfaces that sweep out the manifold, starting with a core of one handlebody and ending at a core of the second. This family is geometric controlled by deforming it to a family of harmonic maps. When the manifold is negatively curved, harmonic maps of genus g surfaces have uniformly bounded area. In the manifolds we construct, the geometry forces small area surfaces to line up with small area cross sections of the manifold. As a result we obtain obstructions to the equivalence of distinct Heegaard splittings.

A *stabilization* of a genus g Heegaard surface is a surface of genus $g + 1$ obtained by adding a 1-handle whose core is parallel to the surface. Such a surface splits the manifold into two genus $g + 1$ handlebodies, and thus gives a new Heegaard splitting. Two genus g Heegaard splittings are *k -stably equivalent* if they become equivalent after k stabilizations. Any two Heegaard splittings become equivalent after a sequence of stabilizations. An upper bound on the number of stabilizations needed to make two splittings equivalent is known in some cases. If G_p and G_q are splittings of genus p and q with $p \leq q$, and M is non-Haken, then Rubinstein and Scharlemann obtained an upper bound of $5p + 8q - 9$

for the genus of a common stabilization. But all previously known examples of manifolds with distinct splittings become equivalent after a single stabilization of the larger genus Heegaard surface. It has been conjectured that a single stabilization always suffices (See Kirby Problem List, Problem 3.89). This is sometimes called the *stabilization conjecture*. We show that this conjecture does not hold. There are pairs of genus g splittings of a manifold that require no fewer than g stabilizations to become equivalent.

Theorem. *For each $g > 1$ there is a 3-manifold M_g with two genus Heegaard g splittings that require g stabilizations to become equivalent.*