Harmonic maps between singular spaces
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The seminal work of M. Gromov and R. Schoen extends the study of harmonic maps between smooth manifolds to the case when the target is a Riemannian simplicial complex of non-positive curvature. The theory of harmonic maps into singular spaces was expanded substantially by the work of N. Korevaar and R. Schoen where they consider targets that are arbitrary metric spaces of non-positive curvature. (Such spaces are called NPC or CAT(0) if they are simply connected.) One important motivation for considering singular spaces in the theory of harmonic maps is in studying group representations. The main application of the Gromov-Schoen theory is to establish a certain case of non-Archemedean superrigidity complementing Corlette's Archemedean superrigidity for lattices in groups of real rank 1.

The next step in the generalization of the harmonic map theory is to replace smooth domains by singular ones. This problem is also motivated by superrigidity, in this case when the domain group is non-Archemedean. The consideration of a Riemannian simplicial complex as the domain space for harmonic maps seems to have been initiated by J-Y. Chen. Subsequently, this theory was further elaborated by J. Eells and B. Fuglede. In particular, they show Hölder continuity for harmonic maps under an appropriate smoothness asumption for the metric on each simplex.

Recall that the main idea of Gromov-Schoen is also to show that harmonic maps are regular enough so that Bochner methods could be used in the setting of singular targets. In particular, the fundamental regularity result of Gromov-Schoen and of Korevaar-Schoen is that harmonic maps from a smooth Riemannian domain into a NPC target are locally Lipschitz continuous. This statement no longer holds when we replace the domain by a polyhedral space. On the other hand, we have found that modulus of continuity better than Hölder is crucial in applications. This necessitates stronger regularity results than Hölder.

In our regularity theorems, we prove that harmonic maps from an admissible n-dimensional Riemannian complex into a NPC space is Lipschitz continuous away from the $(n-2)$-skeleton. Furthermore, near a point on a k-skeleton, we give explicit dependence of the Hölder exponent of a harmonic map on the combinatorial and geometric information of the link of the k-dimensional skeleton. Using this, we provide a sufficient criterion for a harmonic map to be Lipschitz continuous.

The development of the harmonic map theory from a Riemannian complex is important in the study of non-Archedemean lattices. We establish certain fixed point and rigidity theorems of harmonic maps Riemannian complexes. The key issue in the techniques we introduce is to prove regularity theorems strong enough to be able to apply differential geometric methods.