A generalization of Caffarelli's Contraction Theorem via (reverse) heat-flow

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A theorem of L. Caffarelli implies the existence of a map T, pushing forward a source Gaussian measure to a target measure which is more log-concave than the source one, which contracts Euclidean distance. (In fact, Caffarelli showed that the optimal-transport Brenier map is a contraction in this case.) This theorem has found numerous applications pertaining to correlation inequalities, isoperimetry, spectral-gap estimation, properties of the Gaussian measure, and more.

We generalize this result to more general source and target measures, using a condition on the third derivative of the potential. Contrary to the non-constructive optimal transport map, our map's inverse T^{-1} is constructed as a flow along an advection field associated to an appropriately modified heat-flow. The contraction property is then reduced to showing that log-concavity is preserved along the corresponding diffusion, by using a maximum principle for parabolic PDE. In particular, Caffarelli's original result immediately follows by using the Ornstein–Uhlenbeck process and the Prékopa–Leindler Theorem. We thus avoid using Caffarelli's regularity theory for the Monge-Ampère equation, lending our approach to further generalizations. As applications, we obtain new correlation and isoperimetric inequalities.

This is joint work with Young-Heon Kim (UBC).