SPLITTINGS, HYPERBOLIC GEOMETRY AND MODELS

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The question of classifying the closed 3-manifolds is possibly the most important question which has motivated the research in the field of the low dimensional topology. This question saw a major turning point with Thurston's fundamental work in connecting the geometry and topology of 3-manifolds. In particular in his Geometrization Conjecture, Thurston conjectured that a closed 3-manfild can be decomposed (topologically) into pieces which admit one of eight possible geometric structures. He went further and not only proved this for a large class of 3-manifolds but also showed that "most" of these pieces cannot admit any of these geometric structure other than the hyperbolic structure. In particular to solve the conjecture it would be enough to prove these manifolds admit a hyperbolic structure. This remained the most important conjecture in the field until Perelman's recent claim in proving the conjecture using analytic methods.

In our work, we try to emphasize that Perelman's approach and our understanding of his proof lacks an effective construction of the hyperbolic metric. This is one reason why there are still many questions and conjectures regarding the topology of the 3-manifolds which remain unanswered even after assuming the Geometrization. In our work, we try to follow some of Thurston's original ideas and many new developments in the field to obtain a constructive approach to find and describe the hyperbolic metric. We explain how this approach provides answers to questions which remained unanswered even after knowing the Geometrization conjecture is true..

One of the most common ways of constructing 3-manifolds (topologically) is to start from a number of compact 3-manifolds (with boundary) and gluing them along their boundary components. Then by changing the gluing map, one can produce infinite families of such manifolds. A natural question is to ask how the combinatorics of the gluing maps affect the geometry of the final 3-manifold. We introduce a type of combinatorial information on the boundary components of these three manifolds which we call a *decoration* and we say a 3-manifold is *decorated* if every boundary component is equipped with one such decoration. Fixing a finite family of decorated compact 3-manifolds and a constant $R > 0$, we introduce a special type of gluing called *gluing with R-bounded combinatorics*. The introduction of such gluings is motivated by work of Masur-Minsky on the *complex of curves* of surfaces and also by Minsky's work in relating these to the geometry of ends of open hyperbolic 3-manifolds. our main theorem is the following:

Date: Oct 10, 2008.

Theorem 1. *Given a finite family $\mathcal{M}$ of decorated manifolds and $R > 0$ if a manifold X is obtained from copies of elements of $\mathcal{M}$ with "sufficiently complicated" gluings with R-bounded combinatorics, then if X is also hyperbolic we can describe the geometry of X with its hyperbolic metric using a bi-Lipschitz model.*

In order to prove this theorem we use some *convergence theorems* which are generalizations of Thurston's double limit theorem. These however owe more to Morgan-Shalen's approach in proving Thurston's theorem. We describe a generalization of Morgan-Shalen work and then after applying other results by Skora, Namazi-Souto and Kleineidam-Souto we prove our convergence result. We use this convergence results for hyperbolic structures on the interiors of decorated manifolds in $\mathcal{M}$. It follows from those that with a suitable choice of such structures, one can glue these and obtain a negatively curved metric on X whose sectional curvatures are pinched between $-1-\epsilon$ and $-1+\epsilon$ for $\epsilon > 0$ small. Besides the geometry of these metrics can be described using a combinatorial models.

The negatively curved metrics allow us to use the convergence results for the represenations that are induced from a hyperbolic structure on manifolds obtained by R-bounded combinatorial gluings of elements of $\mathcal{M}$. Then using the convergence results we show that the hyperbolic metric is also bi-Lipschitz to the constructed combinatorial model. This finishes our main theorem.

Our main application of the main theorem is for the Heegaard splittings. Given $g > 1$, we use the theorem to introduce a class of splittings which we call *splittings with generalized R-bounded combinatorics*. Then we show that

Theorem 2. *Given R there exists $\epsilon > 0$ such that if a hyperbolic 3-manifoldd admits a genus g splitting with R-bounded combinatorics then its injectivity radii at all points are at least ϵ. Vice versa, given $\epsilon > 0$ there exists R such that if all the injectivity radii at all points of a hyperbolic 3-manifold are at least ϵ and the manifold has a genus g splitting then the splitting must have generalized R-bounded combinatorics. Even more if a hyperbolic 3-manifold satisfies any of the above, it is bi-Lipschitz to the described combinatorial model.*