Dehn surgeries that reduce the Thurston norm of a fibered 3–manifold

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Abstract

Dehn surgery is an important method of constructing 3-manifolds: basically all 3–manifolds can be constructed via it. A general question about Dehn surgery is: how do the invariants of 3-manifolds behave under Dehn surgeries? In this talk, we will consider the behavior of Thurston norm under Dehn surgeries.

Definition 1. Let S be a compact oriented surface with connected components $S_1, \dots, S_n$. We define

$$x(S) = \sum_i \max\{0, -\chi(S_i)\}.$$

Let M be a compact oriented 3–manifold, let $h \in \mathrm{H}_2(M, \partial M)$. The *Thurston norm* $x(h)$ of h is defined to be the minimal value of $x(S)$, where S runs over all the properly embedded surfaces in M with $[S] = h$.

A very important result about Thurston norm is due to Thurston: suppose S is a compact leaf of a taut foliation of a 3–manifold, then S is Thurston norm minimizing in its homology class. Later works of Gabai made this result practical by constructing taut foliations with prescribed compact leaves.

When Y $(\neq S^2 \times S^1)$ is a surface bundle over S^1, the fibration is a taut foliation, and every fiber is a compact leaf of the foliation. So Thurston's theorem says that the fiber of the surface bundle is Thurston norm minimizing in Y.

Here is a simple fact:

Fact. Suppose $F \subset Y$ is a connected Thurston norm minimizing surface, $x(F) > 0$. $K \subset Y$ is a knot on F, and λ is the slope on K specified by F. Let X be the manifold obtained by doing λ–surgery on F, then F (viewed as a surface in X) is compressible in X. Let x_X, x_Y denote the Thurston norm in X and Y, respectively, then

$$x_X([F]) < x_Y([F]).$$

Our main theorem is a converse to the above fact in the case when Y is fibered.

Theorem 2. *Suppose Y is a surface bundle over the circle, and F is a fiber. Let $K \subset Y$ be a knot such that $[K] \cdot [F] = 0$, and let α be a slope on K. Let X be the manifold obtained by α–surgery on K. Since $[K] \cdot [F] = 0$, $[F]$ can also be viewed as a homology class in $H_2(X, \partial X)$.*

Let x_X, x_Y denote the Thurston norm in X and Y, respectively. If

$$x_X([F]) < x_Y([F]),$$

then there is an ambient isotopy of Y which takes K to a curve in F. Moreover, the slope α coincides with the frame on K which is specified by the surface F.

The proof of the above theorem uses Gabai's sutured manifold theory, as well as an argument introduced by Ghiggini in contact topology.