Twisted Alexander polynomials and fibered 3-manifolds

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Abstract: Symplectic 4–manifolds are perhaps the best understood class of symplectic manifolds, but in spite of this fact we still do not have, even at conjectural level, a classification scheme, and are unable to decide in general whether a given 4–manifold admits or not a symplectic structure. A particular class of manifolds where, conceivably, the problems above can be tackled is the class of 4–manifolds that admit a circle action. We focus on a particular case, namely product manifolds of the form $S^1 \times N$, where N is a closed 3–manifold. About 30 years ago William Thurston proved that if N admits a fibration over S^1, then $S^1 \times N$ admits a symplectic structure, and it has long been conjectured that this condition is not only sufficient but also necessary. Seiberg–Witten theory, especially through the work of Cliff Taubes for the symplectic case, provides a tool to decide this question. In collaboration with Stefan Friedl, we have used the information arising from the Seiberg-Witten invariants of $S^1 \times N$ (and all its finite covers) to show that a family of invariants of N, the twisted Alexander polynomials, satisfy rather strict constraints, that where previously known to holds for fibered 3–manifolds. These constraints are the analog of a well-known condition on the Alexander polynomial of a fibered knot, namely that the polynomial is monic and its degree equals the genus of the knot. Led by this result, and numerical evidence, we have conjectured that these constraints completely characterize fibered 3—manifolds.

In the talk I present a proof of this conjecture, obtained in collaboration with Friedl. The strategy can be summarized as follows. First, it is known that the proof can be reduced to the case where N is an irreducible manifold. Denote by Σ a connected minimal genus surface that represent a primitive homology class in $H_2(N)$ for which the constraints are satisfied. Denote by M the exterior of Σ, i.e. $M = N \setminus \nu\Sigma$. Then M is a manifold with boundary given by two copies of Σ. It is well known that, with either inclusion, there is an injective map $\pi_1(\Sigma) \to \pi_1(M)$. We show that the aforementioned constraints imply that this map induces an isomorphism of prosolvable completions. Then, under the assumption that the fundamental group of N is residually finite solvable, we can show that the isomorphism of prosolvable completions actually implies that $M = \Sigma \times I$ (i.e. that N fibers with fiber Σ) by implementing the criterion for virtual fibrations recently determined by Ian Agol. In general, the fundamental group of a 3–manifold is not residually finite solvable, but by going to a suitable finite cover of N we can obtain a property that is close enough to allow one to apply the result above.