The International Whaling Commission Model

A Sperm whale calf investigating the camera!

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PHYSICAL DESCRIPTION  The head of the sperm whale is blunt and squared off, and has a small, underslung jaw. The head is also large, and makes up to 1/3 the total body length and more than 1/3 of its mass. A single blowhole is located forward on the left side of the head, and the blow, which is bushy, is projected forward rather than straight up as it is with other whales. Its body has a wrinkled, shriveled appearance, particularly behind the head.

COLOR  The sperm whale is usually a dark, brownish gray with light streaks, spots and scratches. The skin around its mouth, particularly near the corners, is white. The ventral (underside) of the body is a lighter gray and may have white patches.

FINS AND FLUKES  The sperm whale has a squat dorsal fin, followed by knuckles along the spine. Its flippers are small and slightly tapered, while its flukes are broad, measuring as much as 16 feet (5 m) from tip to tip.

LENGTH AND WEIGHT  Adult males reach lengths of 49-59 feet (15-18 m) and weigh up to 35-45 tons (31,750-40,800 kgs). Adult females are much smaller, growing to about 36 feet (11 m) and a maximum weight of 13-14 tons (12,000-12,700 kg).

FEEDING  Its main source of food is medium-sized deep water squid, but it also feeds on species of fish, skate, octopus, and smaller squid. A sperm whale consumes about one ton (907 kg) of food each day. The lower jaw contains 18-25 large teeth on each side of the jaw, 3-8 inches in length. The upper jaw may have tiny teeth but they rarely erupt. The upper jaw contains a series of sockets into which the lower teeth fit.

MATING AND BREEDING  Males reach sexual maturity at approximately 33-39 feet (10-12 m), and 10 years or more of age but do not seem to take an actual part in breeding until their late 20's. Females reach sexual maturity at 27-29 feet (8.9 m), and 7-13 years of age. Gestation is 14-16 months. Newborn calves weigh approximately 1 ton (907 kg), and are 11-16 feet (3.4-4.9 m) long. Calves nurse up to two years or longer. Contrary to earlier belief, sperm whales do not seem to have harems. Instead, large males only attend female groups a few hours at a time. These female groups (family groups) consist typically of 10-20 animals. Within these groups there appears to be communal care for the young.

The IWC model we’ll be looking at was discussed by Robert M. May, in “Some Mathematical Questions In Biology” volume 13. May starts with a difference equation, then proceeds to a differential equation. We’ll just do the difference equation.
Equilibrium and stability are a little different for difference equations than for differential equations. The first general difference equation we’ll look at is:

\[ x_{n+1} = F(x_n) \]

For example, in the case of exponential growth, \( F(x) = rx \). A logistic kind of growth might be modeled by

\[ F(x) = rx \left( 1 - \frac{x}{K} \right) \]

An equilibrium occurs when \( x_{n+1} = x_n \); this leads to \( F(x_n) = x_n \). So \( x^* \) is an equilibrium point if \( F(x^*) = x^* \).

1) Find all equilibria of the exponential and logistic models above.

An equilibrium is locally stable if . . . well, if a solution near \( x^* \) doesn’t grow exponentially quickly away from \( x^* \). How can we tell this? A close solution would look like

\[ x_n = x^* + cy^n \]

where \( c \) is a small constant, and the solution would return to \( x^* \) if \(|y| < 1\). But how can we tell?

2) Use the Taylor series expansion of \( F \) near \( x^* \),

\[ F(x^* + cy^n) \approx F(x^*) + F'(x^*)cy^n \]

to show that \( x^* \) is locally stable if and only if \(| F'(x^*) | < 1 \).

3) Determine local stability of the exponential and logistic models above.

Unfortunately, whales need more complicated models. After a whale is born, it may take six to eight years before it then contributes to reproducing the species. So \( x_{n+1} \) depends on \( x_n \), but also on \( x_{n-8} \), say. If \( \beta \) is the number of years required for a whale to reach maturity, our equations will look something like:

\[ x_{n+1} = F(x_n) + G(x_{n-\beta}) \]

This is called a delay difference equation and again the analysis of equilibrium and stability is a bit different.
4) Equilibrium, actually, isn't different: it occurs for any $x^*$ with $x^* = F(x^*) + G(x^*)$. Stability, now .... prove that $x^*$ is locally stable if and only if all solutions to the equation

$$y^{\beta+1} - F'(x^*)y^\beta - G'(x^*) = 0$$

satisfy $|y| < 1$. This is a whole bunch harder to determine; if you do a web search on ”Jury stability’ you’ll find some very complicated ways to answer this.

We, folks, are gonna cheat: we’ll develop our equations for whales, then use Matlab to find the roots $y$ above and check if they satisfy $|y| < 1$.

The equations: if $x_n$ is the number of sexually mature whales, we expect an initial term

$$x_{n+1} = (1 - \mu)x_n$$

where $\mu$ is the death rate; this is our $F$. We should also add a term for all the new whales who will come to maturity from $\beta$ years ago. The $G$ this produces is

$$G(x_{n-\beta}) = \frac{1}{2}(1 - \mu)^\beta x_{n-\beta} \left[ P + Q\{1 - \left(\frac{x_{n-\beta}}{K}\right)^z\} \right]$$

What’s going on here? We know that $K$ is a kind of carrying capacity; we’ve even seen he effect of different $z$’s in speeding up or slowing down approach to carrying capacity. Both $P$ and $Q$ are birth rates; the $(1 - \mu)$ is a death rate.

5) What are $P$ and $Q$? Why is $(1 - \mu)$ raised to the $\beta$ power? Why is there a factor of .5 in front?

6) Use equilibrium to prove that $\mu = \frac{1}{2}(1 - \mu)^\beta P$.

7) Use the above to rewrite the equation as

$$x_{n+1} = (1 - \mu)x_n + \mu x_{n-\beta} \left[ 1 + q\{1 - \left(\frac{x_{n-\beta}}{K}\right)^z\} \right]$$

What is $q$ in terms of the previous variables?

8) Show that $x^* = 0; x^* = K$ are the only equilibria.
Now, for stability. Here’s the real-world data:

<table>
<thead>
<tr>
<th>Species</th>
<th>q</th>
<th>z</th>
<th>μ</th>
<th>β</th>
</tr>
</thead>
<tbody>
<tr>
<td>sei whale</td>
<td>0.95</td>
<td>2.4</td>
<td>0.06</td>
<td>8</td>
</tr>
<tr>
<td>minke whale</td>
<td>1.09</td>
<td>2.4</td>
<td>0.095</td>
<td>6</td>
</tr>
<tr>
<td>sperm whale</td>
<td>0.25</td>
<td>2.4</td>
<td>0.055</td>
<td>10</td>
</tr>
</tbody>
</table>

9) For which species are each of the two equilibria stable? Probably the easiest way to do this is to use Matlab or Mathematica; either will give you the roots of the polynomials. Include those in your answer...

10) Now change the equations to account for whaling. Examine the following three scenarios:

a) A constant number $J$ of adults are harvested each year.

b) A constant percentage $h$ of adults and of children are harvested each year.

Under what conditions on $J$ and $h$ are there non-zero, stable equilibria now?