**Introduction:** In a general population growth model, the equations act as though they are like a force, pushing the population to equilibrium – if there is an equilibrium – or to infinity, if there is none. Randomness acts to hamper this motion; it *diffuses* the population, scatters it (as we’ll see when we do the models). It might push a logistic population away from equilibrium; it might keep an exponentially growing population away from infinity.

**Overview:** What we’re going to do is derive an equation for the probability density $P(x)$ of a population having value $x$. The density tells you the probability that you get such and such a population. For example,

$$\int_0^1 P(x) \, dx$$

is the probability that the population is between zero and one. That’s a good way to guarantee extinction! And

$$\int_0^\infty P(x) \, dx$$

is the probability of having some population. Which ought to come out to 1. And

$$\int_0^\infty x \, P(x) \, dx$$

is the average value the population takes on. We’ll find an equation for this density, solve it, find how population growth and diffusion must balance off to prevent extinction, and compute the average population size.

**The equations:** We’ll assume exponential growth, and a random component. The random part will be normally distributed, with mean zero and standard deviation $\sigma^2$. The mean zero part says that the random pushes to the population are just as likely to be up (positive) as down (negative): over the long run, the ups and downs cancel each other out to zero. What I want to show now is the meaning of the statement that the standard deviation is $\sigma^2$. 
1) Let \( f(x) \) be defined as the Gaussian

\[
f(x) = e^{-\frac{x^2}{2\sigma^2}}
\]

a) Plot \( f \) for different values of \( \sigma \): say \( \sigma = 1, 4, 16 \). What does \( \sigma \) control about the graph?

b) Make your answer to a) precise by solving for the inflection points, \( f''(x) = 0 \), and then plotting those \( x \) on the graphs in part 1a.

2) If the differential equation governing the non-random, average growth of the population \( x \) is

\[
\frac{dx}{dt} = M(x)
\]

then the differential equation governing the competition between the random part and the non-random part is:

\[
\frac{d}{dx} \left( M(x)P(x) \right) = \frac{1}{2} \frac{d^2}{dx^2} \left( \sigma^2 x^2 P(x) \right)
\]

(this is called the Fokker-Plank equation and we are not going to derive it!).

a) Show that the equation has solution

\[
P(x) = \frac{C}{\sigma^2 x^2} \exp \left[ 2 \int \frac{M(x)}{\sigma^2 x^2} \, dx \right]
\]

b) Now let \( M \) represent exponential growth: \( M(x) = rx \) where \( r > 0 \). Solve the above for \( P \).

**Analysis:** Now that we know what \( P \) is, we want to see what it tells us about the population growth.

3) Show that the extinction probability,

\[
\int_0^1 P(x) \, dx
\]
is finite if and only if $2r > \sigma^2$. Since probabilities can’t be infinite, we have to assume this from now on.

4) We don’t know what $C$ is. We usually get this by requiring that the total probability is one:

$$\int_{0}^{\infty} P(x) \, dx = 1$$

Show this can never happen if $2r > \sigma^2$. Why is this reasonable?

5) Let’s assume instead that the greatest value the population can have is some number $N$. Solve for $C$ in the equation

$$\int_{0}^{N} P(x) \, dx = 1$$

6) Using the value of $C$ just found, compute the extinction probability

$$\int_{0}^{1} P(x) \, dx$$

and show that as $2r - \sigma^2 \to 0$, the probability of extinction approaches 1. What does this mean, biologically?

7) Using the answer to part 6, assume the population limit $N$ is $N = 10$. What value must $\frac{2r}{\sigma^2} - 1$ be so that the probability of extinction is less than .5? Repeat the computation for $N = 1,000,000$. What do you conclude?

8) Compute the average population,

$$\int_{0}^{N} x \, P(x) \, dx$$

What does this show the effect of randomness is? As part of your answer, discuss the limiting cases $\sigma^2 \to 2r; \sigma^2 \to 0$. 