Section 3.2: Absolute Extremes

The picture really does tell the story: I’m looking for the extremes: highest and lowest. On the surface of the earth, the highest point above sea-level is Everest. If you measure height by distance from the center of the earth, Chimborazo in Ecuador is taller. We’d like to avoid all these problems.

Definition Let \( y = f(x) \) be a function defined on an interval \( I \). We say \( f \) has an absolute maximum at the point \( c \) if \( f(x) \leq f(c) \) for all \( x \) in \( I \). Also, \( f \) has an absolute minimum at \( c \) if \( f(x) \geq f(c) \) for all \( x \) in \( I \). Together, these are called absolute extremes. Our goal is to locate \( c \). This is tricky, because as \( x \) varies, \( f \) takes on an infinite number of different heights, so we can’t check them all.

What I want is someone to make a ‘short-list’ for me, like the list in Figure 2. It’s a list of investments, along with the percentage increase you might expect each year. It’s a small list, so you could just scan through and find the best one. That’s what I want for my absolute extremes: a list of candidates. Here it is:

Theorem If \( f(x) \) is a continuous function defined on \( I = [a, b] \), then \( f \) has absolute extremes on \( I \). There are two kinds of candidates:

i) Interior points \( c: a < c < b \) and \( f'(c) = 0 \) or \( f'(c) \) does not exist

ii) Boundary points \( c: c = a \) or \( c = b \).

Every extreme is one of these. Let’s see some examples, then move on to see why these are the only candidates.
We’ll start with the least complicated example we can find, Figure 3, showing \( f(x) = x \) on \([-1, 1]\). You can check if you like, but the absolute extremes are at the endpoints, \( \pm 1 \). Here’s the check: for \( x \) to be in \( I \) means \(-1 \leq x \leq 1\). But \( f(x) = x \), so we can just rewrite that as \( f(-1) \leq f(x) \leq f(1) \). Which shows \( f \) has largest value at \( x = 1 \), and smallest value at \( x = -1 \).

Okay, I admit: that’s really simple, but what we learn from this is that extremes can occur at the ends of an interval. These are called boundary extremes, and now we know that if \( f \) is defined on the closed interval \([a, b]\), then \( a \) and \( b \) have to included in our short list of candidates.

Here’s another easy yet helpful example: Figure 4, \( f(x) = x^2 \) on the open interval \((-1, 1)\). Because the interval is open, \( f \) gets higher and higher as \( x \) goes to \( \pm 1 \), but can never attain the highest value. So \( f \) has no absolute maximum. But \( x^2 \geq 0 \) always, or, in \( f \) language, \( f(x) \geq f(0) \): that is, \( f \) has an absolute minimum at \( x = 0 \).

So what profound truths do we learn here? This time, the extreme is inside the interval; such an extreme is called an interior extreme. Moreover, at this extreme, the derivative is zero.

Figure 5 complements Figure 4: it’s the curve \( f(x) = x^{\frac{3}{2}} \) on the closed interval \([-1, 1]\), and again I’m interested in the interior extreme at \( x = 0 \), where the derivative fails to exist.

Every picture tells a story; the story here is that interior extremes can occur at critical points. But the absolutely amazing thing is that boundary points and interior critical points give me a complete list of candidates:

**Theorem** If \( f \) is continuous on \( I = [a, b] \) then \( f \) has absolute extremes. Moreover, they are either boundary points or interior critical points.

**Proof** Figure 6 illustrates the idea of the proof. Let’s say \( f \) has an absolute maximum at \( c \), and \( c \) isn’t a boundary point; then it’s interior and the Theorem asks me to show that \( c \) is a critical point. But for \( x < c \), \( f(x) < f(c) \), hence \( f'(x) < 0 \) on \((a, c)\). Similarly, if \( x > c \), \( f(x) > f(c) \), hence \( f'(x) < 0 \) on \((c, b)\). By the intermediate value theorem, we have to have \( f'(c) \) is zero or discontinuous at \( c \). For our functions, that means \( f'(c) = 0 \) or \( f'(c) = \text{dne} \).
So here’s what I learn: to find the absolute extremes of $f$ on $[a, b]$, I need to take the endpoints $a$, $b$, then the critical points, $c_1$, $c_2$, ... and put them in a table, with the ends and critical points in the first column, and the values of $f$ in the second:

<table>
<thead>
<tr>
<th>candidates $x$</th>
<th>values $f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$f(a)$</td>
</tr>
<tr>
<td>$b$</td>
<td>$f(b)$</td>
</tr>
<tr>
<td>$c_1$</td>
<td>$f(c_1)$</td>
</tr>
<tr>
<td>$c_2$</td>
<td>$f(c_2)$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

That’s my short-list. Then the largest number in the second column is the absolute maximum; the smallest the absolute minimum.

**Example** Let $f(x) = x^2(x-1)^3$ on the interval $[-1, 2]$. (See Figure 7).

i) Differentiate and simplify:

$$f'(x) = \left[ x^2 \right]'(x-1)^3 + x^2 \left[ (x-1)^3 \right]' = 2x(x-1)^3 + x^2 \left[ 3(x-1)^2 \right]$$

$$= x(x-1)^2 [2(x-1) + 3x] = x(x-1)^2 (5x - 2)$$

ii) Find all critical points: $f'$ exists for all $x$, so all we have to look for is $c$ where $f'(c) = 0$. Those occur when the factors are zero, that is: $x = 0, x = 1, x = 2/5$.

iii) Make a table of values and locate extremes:

<table>
<thead>
<tr>
<th>candidates $x$</th>
<th>values $f(x) = x^2(x-1)^3$</th>
<th>extreme</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1$</td>
<td>$(−1)^2(−1−1)^3 = −8$</td>
<td>amin</td>
</tr>
<tr>
<td>$2$</td>
<td>$(2)^2(2−1)^3 = 4$</td>
<td>amax</td>
</tr>
<tr>
<td>$0$</td>
<td>$(0)^2(0−1)^3 = 0$</td>
<td>neither</td>
</tr>
<tr>
<td>$1$</td>
<td>$(1)^2(1−1)^3 = 0$</td>
<td>neither</td>
</tr>
<tr>
<td>$2/5$</td>
<td>$(2/5)^2(2/5−1)^3 = ?$</td>
<td>neither</td>
</tr>
</tbody>
</table>

That last line – $(2/5)^2(2/5−1)^3 = ?$. Really? That’s the best I can do? I have my usual hidden agenda: the number in question is a fraction. So, whatever the real value ($-108/3125$), it’s a fraction; its value $v$ satisfies $−1 < v < 1$. The competition are numbers like 4 and -8; this little fraction never had a chance. Sad. But I don’t need to know more about it, than that it is a fraction.
Before we go on to more examples, we’ll step back, ask our favorite question: ‘who cares?’ It’d be like if you don’t watch tennis; you don’t care that Serena Williams is the best in the world, or whoever is worst (just for the record: atashi).

So let’s meet people who design drugs (the legal kind). Figure 8 shows what they deal with. You swallow a drug; it goes through your stomach and intestines, is released into the bloodstream. Figure 8 shows the drug concentration building up to a peak, as more and more drug is released. At the same time, the liver and kidneys break down, filter out and eliminate the drug, acting to decrease the concentration. The peak is a compromise between release into blood, and elimination from blood. As we’ve said before: critical points often are transitions from one region to another.

Since release and elimination happen at the same time, the peak concentration will be less than the amount you actually took.

The question drug designers need to ask is whether the blood concentration will be enough to help you; they also have to worry if the concentration will be too much, and whether the drug will harm you. For any drug, there’s a region: the safe and effective region, bounded by two red lines in Figure 8.

The dose you take determines the blood concentration; it’s prescribed so that the peak is above the effective line; it’s also prescribed so the peak is below the safe line. So the location of the absolute maximum matters a lot!

The next example is from Engineering. Take a regular sheet of notebook paper; grab the ends with your hands, and pull. Odds are, the paper doesn’t tear. Figure 9 shows why: the paper fibers redistribute the force throughout the whole sheet. So the force you use gets diluted over the entire sheet, like a drop of milk in a gallon of coffee. Diluted, it’s too small to separate the fibers and: no tearing.

For part two of the experiment, you cut a very small, very thin line in the middle of the paper, as in Figure 10. This time, the paper tears apart almost immediately, even with just a little force. What’s going on, here?
This is about more than just paper. When transatlantic ships began to be built with iron instead of just wood, ship designers used much the same kinds of designs, and the ships were as reliable, even in storms, as wooden ships. You’d expect that: iron ought to be stronger than wood! But, when the elevator was invented, a square hole was cut in the deck of a ship, for the elevator shaft. The first voyage out, the ship broke in two and sank. Again: What’s going on, here? With the ship and the paper, a small cut reduces the strength of the material to nothing.

Figure 11 shows what’s happening; it’s a graph of stress, over the cut, as the paper/metal is pulled. Notice the peaks, at the ends of the cuts: most of the force you use to pull the paper gets redistributed. This time, the force doesn’t get diluted over the whole paper; it gets concentrated at the cut. And that very concentrated force is plenty to separate the fibers, tear the paper, break the metal.

(Why? You can can think about it with fibers, as in Figure 9. Right at the tip of the cut, the paper would like to pass the stress on to the next fiber. But there’s a cut! There aren’t any fibers to take the stress! So the stress builds up, can’t be transferred. Boom.)

Example That was kind of fun, but it’s back to the computations, this time $y = \cos^2 x + \sin x$ on the interval $[-\pi, \pi]$, shown in Figure 12.

i) Differentiate and simplify:

$$f'(x) = [\cos^2 x]' + [\sin x]' = [2 \cos x (-\sin x)] + [\cos x] = \cos x [1 - 2 \sin x].$$

ii) Find all critical points: As before, $f'$ exists so we check for where $f'(c) = 0$. Those occur when $\cos x = 0$ and when $\sin x = \frac{1}{2}$. These are easy with a calculator, but I’ll use the unit circle.

For $\cos x = 0$, cosine represents a horizontal co-ordinate on the unit circle, so $\cos x = 0$ when the horizontal co-ordinate is zero. You can see that by taking the line $x = 0$ and intersecting it with the circle, as we did in Figure 13. The line hits the circle at $\pm \frac{\pi}{2}$.

For $\sin x = \frac{1}{2}$, sine represents a vertical co-ordinate on the unit circle, so $\sin x = \frac{1}{2}$ when the vertical co-ordinate is a half. You can see that by taking the line $y = \frac{1}{2}$ and intersecting it with the circle, as we did in Figure 14. This time, the angles aren’t so clear: you have to know something extra. That something is that $\sin x = \pm \frac{1}{2}$ at multiples of $\frac{\pi}{6} = 30^\circ$. Looking at Figure 14, I can see the intersection in the first quadrant is thirty degrees rather than, say, sixty; the intersection in the second quadrant is pretty clearly then $180^\circ - 30^\circ = 150^\circ = \frac{5\pi}{3}$. Thus, solutions $\frac{\pi}{6}, \frac{5\pi}{6}$.  

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**Figure 11:** Stress Distribution
Now we start to pull on the sides of the paper. The force is distributed through the place, but the peak show where the force is greatest: right at the cut.

**Figure 12:** A Trig Example
The graph of $y = \cos^2 x + \sin x$. The blue dots tell me candidates for extremes.

**Figure 13:** A Trig Equation
Solving the (easy) equation $\cos(x) = 0$, using the unit circle.

**Figure 14:** Another Trig Equation
Solving $\sin(x) = \frac{1}{2}$, again with the circle.
iii) Make a table of values and locate extremes:

<table>
<thead>
<tr>
<th>candidates $x$</th>
<th>values $f(x) = \cos^2 x + \sin x$</th>
<th>extreme</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\pi$</td>
<td>$\cos^2(-\pi) + \sin(-\pi) = 1 + 0$</td>
<td>$= 1$ neither</td>
</tr>
<tr>
<td>$\pi$</td>
<td>$\cos^2(\pi) + \sin(\pi) = 1 + 0$</td>
<td>$= 1$ neither</td>
</tr>
<tr>
<td>$-\frac{\pi}{2}$</td>
<td>$\cos^2(-\frac{\pi}{2}) + \sin(-\frac{\pi}{2}) = 0 - 1$</td>
<td>$= -1$ amin</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>$\cos^2(\frac{\pi}{2}) + \sin(\frac{\pi}{2}) = 0 + 1$</td>
<td>$= 1$ neither</td>
</tr>
<tr>
<td>$\frac{\pi}{6}$</td>
<td>$\cos^2(\frac{\pi}{6}) + \sin(\frac{\pi}{6}) = \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2}$</td>
<td>$= \frac{5}{4}$ amin</td>
</tr>
<tr>
<td>$\frac{5\pi}{6}$</td>
<td>$\cos^2(\frac{5\pi}{6}) + \sin(\frac{5\pi}{6}) = \left(-\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2}$</td>
<td>$= \frac{5}{4}$ amin</td>
</tr>
</tbody>
</table>

I want took at a few more applications of the idea of extremes; the first will be from physics. Figure 15 shows light bending through a prism. Hero of Alexandria (60 C.E.) explained multiple reflections in mirrors by saying that light travelled in the shortest possible path. Ibn al-Haytham (1021 C.E.) understood that light travelled slower in glass than in air, and explained bending of light by a prism by saying that light travelled to minimize the time to go between two points.

Since the law explaining the bending of light (Snell’s law, discovered by Ibn Sahl 984 C.E.) can be checked experimentally, these ‘least distance, least time’ laws seem a little pointless. So let’s kick this up a notch: how do we explain the soap bubble in Figure 16? It turns out this too is a ‘least’ object: the soap bubble is the surface with least area containing a certain volume. You can fix the volume by forcing the bubble to hang on wires; the bubble then falls into the shape with smallest area.

Here’s another: what’s the shortest curve connecting two points? We all know it’s a line, but how do we know? I mean, have you checked all the other curves (this one, we’ll actually answer in Volume III).

Again, you may not find this very impressive, but the French physicist-astronomer-mathematician Lagrange (born 1736) rewrote Newton’s laws, explaining motion under forces as mimima of certain functions, now called Lagrangians. Figure 17 shows the Lagrangian for the Standard Model: the physical theory uniting electrons, protons, quarks, and even the Higgs Boson, all in one (big) expression. Now that’s an extreme!

Now an application from biochemistry: enzymes. Most of the chemical reactions in a cell, in the human body, go nowhere at all, without extra help. The classic example is a packet of sugar. In your body, the sugar gets converted to water and carbon dioxide and energy. Leave some sugar alone, it just sits there, doing nothing; certainly no $H_2O$, $CO_2$. What’s the difference?
Enzymes. We won’t go into the chemistry, but enzymes change the energy barriers that slow reactions down (sugar left alone) or speed reactions up (sugar in the body). As our guys say it, “The fastest known reactions include reactions catalyzed by enzymes, but the rate enhancements that enzymes produce had not been fully appreciated until recently. In the absence of enzymes, these same reactions are among the slowest that have ever been measured, some with half-times approaching the age of the Earth” (Wolfenden and Snider, Acc. Chem. Res. 2001, 34, 938-945).

Life depends on enzymes. Sounds pompous, but follow the opposite: messing with enzymes messes with life. For example, the drug ibuprofen inhibits the enzyme cyclooxygenase, which mediates inflammation, and fever. Arsenic inhibits pyruvate dehydrogenase, which can inhibit mitochondria from generating energy. Which kills you. So it’s in everyone’s interest to keep their enzymes happy. Vitamins, actually, are one way to do that.

Another way to is regulate the acid-base ratio in the body. Figure 18 shows the effectiveness of a generic enzyme, plotted against the pH of its surroundings. What you see in the figure is that the enzyme is most effective in a very narrow range of pH; after that, enzyme effectiveness drops off sharply. This isn’t surprising: excess ions can bind with the enzyme, changing the shape of the molecule and inhibiting its ability to bind to other molecules.

Thus, the body has elaborate buffering systems to maintain pH, but these can break down. In acidosis, when blood pH drops below 7.0, the central nervous system is depressed; coma and death follow. Diabetes sufferers die in a coma because of this. Alkalosis, in contrast, excites the peripheral nervous system, leading to muscle rigidity and spasms. This can cause death if muscles associated with breathing are affected.

Last example: the heartbeat. The heart is like a fuel-injection engine: the atria collect blood, then contract to inject blood into the ventricles. When the major muscles around the ventricles contract, they provide the main pumping action of the heart. Muscle contraction generates electrical currents; by placing electrodes on the skin, the voltages can be measured. Willem Einthoven was the first to devise a practical measurement technique (see Figure 19); he was awarded the 1924 Nobel Prize for Medicine or Physiology for his work (note: ‘practical’ is the important term here; before his work, heart measurements were either inaccurate, or had to be made by operating to expose the heart).
What you see in Figure 21 is a number of peaks and valleys; the initial contraction of the aorta is a small bump called the P-wave. The ventricular contraction produces the much larger peak, the R-wave. This is usually taken as the reference point for when the heart has pumped, so it’s this absolute maximum that MD’s want to detect. We return to the Introduction: Figure 22, showing a hand-held device to detect the R-wave, and compute quantities like beats-per-minute.

So the question is, how does the little heartpod know where the R-wave is? That’s actually a very complex problem, as beats can be very irregular. But the beginning of the problem isn’t hard at all: look at Figure ?? The presence of the maximum is suggested by the change in slope of the secant lines. Let’s go back to Pan and Tompkins, IEEE Transactions On Biomedical Engineering, BME-232(3), March 1985: A Real-Time QRS Detection Algorithm. They comment, ”The slope of the R wave is a popular signal feature used to locate the QRS complex in many QRS detectors.”

What we did in this chapter looks simple. It is simple, but it is the basis for many, many complex and important applications.

14U Problems

1) Let \( f(x) = \sin^2 x + 2 \sin x \) on \([0, 2\pi]\).
   a) Differentiate and Simplify.
   b) Find all critical points.
   c) Find absolute extremes.

2) Let \( f(x) = x^3(x - 1)^2 \) on \([-1, 2]\).
   a) Differentiate and Simplify.
   b) Find all critical points.
   c) Find absolute extremes.

3) Let \( f(x) = (\ln x)^2 \) on \([e^{-3}, e^3]\).
   a) Differentiate and Simplify.
   b) Find all critical points.
   c) Find absolute extremes.

4) Let \( f(x) = x^2e^{-x^2} \) on \([-2, 2]\).
   a) Differentiate and Simplify.
   b) Find all critical points.
   c) Find absolute extremes.
14U Solutions

1) Let $f(x) = \sin^2 x + 2 \sin x$ on $[0, 2\pi]$.
   a) $2 \cos x [\sin x + 1]$.
   b) $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}$.
   c) $\frac{\pi}{2}$ amax; $\frac{3\pi}{2}$ amin.

2) Let $f(x) = x^3(x - 1)^2$ on $[-1, 2]$.
   a) $x^2((x - 1)(5x - 3))$.
   b) $0, 1, \frac{3}{5}$.
   c) -1 amin; 2 amax.

3) Let $f(x) = (\ln x)^2$ on $[e^{-3}, e^3]$.
   a) $2 \ln x / x$.
   b) 1.
   c) $e^3, e^{-3}$ amax; 1 amin.

4) Let $f(x) = x^2 e^{-x^2}$ on $[-2, 2]$.
   a) $2xe^{-x^2} [1 - x^2]$.
   b) 0, ±1.
   c) 0 amin, ±1 amax.