Alternating Series

What: \( \sum (-1)^k a_k \) or \( \sum (-1)^{k+1} a_k \) \( a_k \geq 0 \)

Example:

\[
\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = (1) \frac{1}{1} \frac{1}{2} + \cdots
\]

\( = 1 - \frac{1}{2} + \frac{1}{3} - \cdots \)

Whereas

\[
\sum_{k=1}^{\infty} \frac{a_k}{k} = (\frac{1}{1} \frac{1}{2} + \cdots) \frac{1}{3}
\]

\( = \frac{1}{1} + \frac{1}{2} - \frac{1}{3} + \cdots \)

Notice:

\[
\sum_{k=1}^{\infty} (-1)^k a_k = a_1 - a_2 + a_3 - a_4 + \cdots
\]

\[
\sum_{k=1}^{\infty} (-1)^{k+1} a_k = -a_1 + a_2 - a_3 + \cdots
\]

\[
= -\left[ a_1 - a_2 + a_3 - \cdots \right]
\]

\( = -\sum_{k=1}^{\infty} (-1)^k a_k \)

So \( \sum (-1)^k a_k \) and \( \sum (-1)^{k+1} a_k \) are

negatives of each other, so they both do the same thing - so they converge or diverge. So sometimes I'll just write about one of them, because the other some plan ahead.
Why?

\[ e^x (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \]

\[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \]

\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \]

All the cool guys are alternating!

(Seriously, if I ever hope to understand \( \ln, \cos \) and \( \sin \), I have to know about alternating!)

So? Why alternate? Let's look at \( \sum \frac{1}{k} \) vs \( \sum \frac{1}{k^2} \)

<table>
<thead>
<tr>
<th># of terms added</th>
<th>( \sum \frac{1}{k} )</th>
<th>( \sum \frac{1}{k^2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>2.929</td>
<td>0.6456</td>
</tr>
<tr>
<td>100</td>
<td>5.187</td>
<td>0.6882</td>
</tr>
<tr>
<td>1,000</td>
<td>7.485</td>
<td>0.6926</td>
</tr>
<tr>
<td>10,000</td>
<td>9.787</td>
<td>0.6931</td>
</tr>
</tbody>
</table>
Let me do the same thing with $\frac{1}{\sqrt{k}}$

<table>
<thead>
<tr>
<th>number of terms added</th>
<th>$\sum \frac{1}{\sqrt{k}}$</th>
<th>$\sum 1/k^{1/2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>5.021</td>
<td>45.07</td>
</tr>
<tr>
<td>100</td>
<td>18.59</td>
<td>55.50</td>
</tr>
<tr>
<td>1,000</td>
<td>61.801</td>
<td>58.91</td>
</tr>
<tr>
<td>10,000</td>
<td>198.5</td>
<td>59.99</td>
</tr>
</tbody>
</table>

\[\sum \frac{1}{\sqrt{k}} \text{ goes to } \infty \]

\[\text{Clearly does not go to } \infty\]

\[\text{Clearly } \sum \frac{1}{kn^{1/2}} \text{ converges}\]

That's what alternating series can do for you.

It can make divergent series into convergent series.

Why? How? What?

Let's try to understand this by looking at $\sum (-1)^{k+1}/k$. 

\[\sum (-1)^{k+1}/k \]

\[
\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \\
= (1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{5} - \frac{1}{6}) + \cdots \\
= \frac{1}{2} + \frac{1}{12} + \frac{1}{30} + \cdots
\]

So compare this with \( \sum \frac{1}{k} \)

\( \frac{1}{2} + \frac{1}{12} + \frac{1}{30} + \frac{1}{42} + \frac{1}{56} + \frac{1}{72} + \frac{1}{90} + \cdots \)

\[ \sum \frac{1}{k} \]

How (0+1) more stuff than

\[ \sum \frac{(-1)^{k+1}}{k} \quad \text{so} \quad \sum \frac{1}{k} \text{ is bigger.} \]

Why does this work? Because \( \sum \frac{(-1)^{k+1}}{k} \) has subtractions and subtractions cancel number and make them smaller.

Small enough to be finite instead of infinite?

So - what does \( b \) have to do in order for \( \sum c_m \) on to converge?
As alternating enough to make something converge?  No - try this

\[ \sum (-1)^k k \]

\[ = 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + \cdots \]

\[ = 1 + (-2 + 3) + (-4 + 5) + (-6 + 7) + \cdots \]

\[ = 1 + 1 + 1 + 1 + 1 + \cdots = \infty \]

We saw this before in class - if \( \lim a_k \neq 0 \)

\[ \text{then} \lim (-1)^k a_k \text{ does not exist} \]

\[ \sum (-1)^k a_k \text{ does by d. t. r.} \]

We checked this with

\[ \sum (-1)^k 1 \text{ geom series } |r| = 1 > 1 \]

\[ \sum (-1)^k 2k \text{ geom series } |r| = 2 > 1 \]

So Th d.w.

If \( \sum (-1)^k a_k \) lies any hope of converging,

we must have \( \lim a_k = 0 \).

Is that enough?
Kathy's Alternating Series Test

If you have $\sum (-1)^n c_n$ or $\sum (-1)^{k+1} c_n$

and $c_n > 0$

and $\lim_{n \to \infty} c_n = 0$

then $\sum (-1)^n c_n$ converges

Unfortunately, Kathy's Test is wrong.

Try this: $(\frac{2}{1} - \frac{1}{1}) + (\frac{2}{2} - \frac{1}{2}) + (\frac{2}{3} - \frac{1}{3}) + (\frac{2}{4} - \frac{1}{4}) + (\frac{2}{5} - \frac{1}{5})$

= $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$

always $p$-test $p=1 > 1$

why didn't this work? After all, everything cancelled.

Heres why gold at $\frac{2}{3} - \frac{1}{3}$ cancels great

but the next term is $+\frac{2}{4}$, I just added

\( \frac{2}{4} \) which is more than I cancelled with the

$-\frac{1}{3}$.

Somewhere: $\frac{2}{4} - \frac{1}{4}$ cancels great

but I add on $\frac{2}{5}$ which is more than $\frac{1}{4}$. 
So this is messed up because I took back more than I took away.

I have to not do this. If I take away a_n, the next term can't be larger, which is a_n+1, can't be larger.

I have to show the next-term smaller.

\[ a_{n+1} \leq a_n \]

New improved K-theory test, usually called Leibniz's test, or the alternating series test.

Assume you have \( \sum (-1)^n a_n \) or \( \sum (-1)^n k n a_n \)

Assume \( a_n \geq 0 \)

Assume \( \lim_{x \to \infty} a_n = 0 \)

Assume \( a_{kn} \leq a_n \) for all \( k \)

Then \( \sum (-1)^k a_n \), \( \sum (-1)^{k+1} a_k \) converge.
Simple example \[ \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \]

1. \( a_k = \frac{1}{k} \geq 0 \)? Yes, \( k = 1, 2, \ldots \) \( \checkmark \)

2. \( \lim_{k \to \infty} a_k = 0 \)? Yes \( \lim_{k \to \infty} \frac{1}{k} = \frac{1}{\infty} = 0 \) \( \checkmark \)

3. \( a_k \leq a_l \)? \( \frac{1}{k} \leq \frac{1}{l} \)? \( \forall k \leq l \)? \( \checkmark \)

So \( \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \) converges by alt. series

Slightly more complicated example

\[ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2 k} \]

1. \( a_k = \frac{1}{k^2 k} \geq 0 \)? Yes—in practice for us, you don't have to bother testing this.

2. \( \lim_{k \to \infty} \frac{1}{k^2 k} = 0 \)? \( \lim_{k \to \infty} \frac{1}{k^{2+1}} = \lim_{k \to \infty} \frac{1}{k^3} = 0 \cdot 0 = 0 \) \( \checkmark \)
2) \( a_{kn} \leq a_k \) ?

\[ \frac{1}{(kn)^{2^k}} \leq \frac{1}{k^{2^k}} \quad \text{or} \quad k \cdot 2^k \leq (kn) \cdot 2^k ? \]

divide both sides by \( 2^k \)

\[ k \leq (kn)(2) ? \]

let \( k \leq 2k + 2 \)

so subtract \( k \)

\[ 0 \leq k + 2 \quad \text{yes,} \]

remember \( k \geq 1 \) always ok

\[ k \leq 1 \] always bad

This is getting sneakier — it's getting harder to check \( a_{kn} \leq a_k \)

But you still have to check. You can't say

\[ k = 1 \quad \Rightarrow \quad k + 1 = 2 \]

\[ \frac{1}{2 \cdot 2^2} \leq \frac{1}{1 \cdot 2^1} ? \]

\[ \frac{1}{8} \leq \frac{1}{2} \checkmark \]

**This is not enough!!**

You have to check \( a_k \) for \( k \), which means you have to do algebra.