Well be looking at indeterminate forms, which are kinds of limits, and before we do that I thought we should talk about what determinate forms are, or, for that matter, what forms are.

The limits well be looking at are mostly like

$$\lim_{x \to 0} \frac{1}{x^2} = \infty \quad \text{or} \quad \lim_{x \to \infty} e^x = \infty$$

The intuition for these is easy; \( \lim_{x \to c} f(x) = \infty \) is supposed to mean that as \( x \) and \( c \) agree to more and more decimal places, \( f(x) \) gets larger and larger.

‘when \( x \) gets close to \( c \), \( f(x) \) gets larger and larger.’ What does that actually mean? Two issues:

i) ‘\( x \) gets close to \( c \)’ refers to all \( x \) close to \( c \); you can’t take one \( x \) where \( f(x) \) is large, and then for the next one, \( f(x) \) is suddenly smaller. \( f \) needs to get large, and stay large.

ii) ‘\( f(x) \) gets larger and larger’ – when we say this, we don’t mean something like \( f \) taking on values like 1, 1, 1, 1.12, 1.123, 1.1234, .... We want \( f \) to be getting infinitely large! One way to measure this is with number of decimal places: we could say, ‘as \( x \) gets close to \( c \), \( f(x) \) has more and more places to the right of the decimal point. As example would be 1.1, 11.12, 111.123, 1111.1234 ... . Tens, hundreds, thousands – \( f \) is forced to get large.

The same intuition works for \( \lim_{x \to \infty} f(x) = \infty \): as \( x \) gets large, \( f(x) \) gets large. So we get very unsurprising results like

$$\lim_{x \to \infty} x = \infty$$

What determinate forms do is try and build up from the simple limit \( \lim_{x \to \infty} x = \infty \) to more complicated limits, like \( \lim_{x \to \infty} x^3 - x^2 + 1 \), and build up here means that I want to do these more complex limits using algebra. One of the very basic limits is this:

If \( \lim_{x \to c} f(x) = \infty \) and \( \lim_{x \to c} g(x) = \infty \)

Then \( \lim_{x \to c} (f(x) + g(x)) = \infty \)

This not only makes sense – if both \( f \) and \( g \) get large, then \( f + g \) has to be even larger – but it wouldn’t be hard to prove using the ideas above.

There’s even more here, though: this is not one limit, like, say, \( \lim_{x \to \infty} x^2 = \infty \), it’s a whole collection of limits, one limit for each function \( f \) and \( g \). This is what the phrase ‘determinate form’ means, and we’re going to write it in a very algebra-looking way,

$$\infty + \infty = \infty$$

but we have to know, in the back of our minds, that this isn’t algebra in the same way \( x + y = y + x \) is algebra; instead, \( \infty + \infty = \infty \) means

If \( \lim_{x \to c} f(x) = \infty \) and \( \lim_{x \to c} g(x) = \infty \)

Then \( \lim_{x \to c} (f(x) + g(x)) = \infty \)

With that in the background, let’s list the determinate forms; in the list, \( \alpha \) means a finite constant like 2, -5, 3.145.
\[ \infty + \alpha = \infty \]
\[ \alpha \cdot \infty = \]
\[ = \infty \text{ if } \alpha > 0 \]
\[ = -\infty \text{ if } \alpha < 0 \]
\[ \infty \cdot \infty = \infty \]
\[ -\infty \cdot \infty = -\infty \]
\[ \frac{1}{\pm \infty} = 0 \]
\[ \infty^\alpha = \]
\[ = \infty \text{ if } \alpha > 0 \]
\[ = 0 \text{ if } \alpha < 0 \]
\[ \frac{1}{0^+} = \infty \]
\[ \frac{1}{0^-} = -\infty \]
\[ e^\infty = \infty \]
\[ e^{-\infty} = 0 \]
\[ \ln(0^+) = -\infty \]
\[ \ln(\infty) = \infty \]
One remark: before we actually start using these, there are the only allowed determinate forms in the class; you can’t make up extra forms by yourself. Lots of people want to; they want to write $2^\infty = \infty$ and especially $0^\infty = \infty$. These aren’t allowed; any other determinate form you want to use, you have to get from the list. Here’s how.

Let’s take, say, $e^{-\infty} = 0$, which we already know. But,

$$e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0$$

This is how we can use one determinate form to get another. You can also use the forms to do limit computations. Here’s a really basic example:

$$\lim_{x \to \infty} (x^2 + 3x + 2) = \infty^2 + 3 \cdot \infty + 2 = \infty + \infty + 2 = \infty + 2 = \infty$$

This is a little more complicated;

$$\lim_{x \to \infty} (x^2 - 3x + 2) = \infty^2 - 3 \cdot \infty + 2 = \infty - \infty + 2 = \infty - \infty$$

except that $\infty - \infty$ isn’t one of our determinate forms, so I have to be smarter:

$$\lim_{x \to \infty} (x^2 - 3x + 2) = \lim_{x \to \infty} x(x - 3) + 2 = \infty \cdot (\infty - 3) + 2 = \infty \cdot \infty + 2 = \infty + 2 = \infty$$

This next is a typical example:

$$\lim_{x \to \infty} \frac{x^2 - 3x + 2}{x^2 + 3x + 2} = \frac{\infty}{\infty}$$

and again, this isn’t a determinate form, so, again, we use algebra to re-write this into determinate forms. In the case of a quotient of polynomials, you can divide numerator and denominator by the highest power you find in the polynomial, in this example, $x^2$:

$$\lim_{x \to \infty} \frac{x^2 - 3x + 2}{x^2 + 3x + 2} = \lim_{x \to \infty} x^2 - 3x + 2 \cdot \frac{1}{x^2}$$

$$= \lim_{x \to \infty} \left( \frac{1 - \frac{3}{x} + \frac{2}{x^2}}{1 + \frac{3}{x} + \frac{2}{x^2}} \right) = \frac{1 - \frac{3}{\infty} + \frac{2}{\infty^2}}{1 + \frac{3}{\infty} + \frac{2}{\infty^2}}$$

$$= \frac{1 - 3 \cdot 0 + 2 \cdot 0}{1 - 3 \cdot 0 + 2 \cdot 0} = \frac{1}{1} = 1$$

Finally, something a little different; I like to give this on exams because it surprises people.

$$\lim_{x \to 0^+} \frac{\ln(x)}{x} = \frac{-\infty}{0^+}$$

Well, that’s not one of my determinate forms, but a tiny bit of algebra can rewrite it:

$$\lim_{x \to 0^+} \frac{\ln(x)}{x} = \lim_{x \to 0^+} (\ln(x)) \left( \frac{1}{x} \right) = (-\infty) \left( \frac{1}{0^+} \right) = (-\infty)(\infty) = -\infty$$