14u Improper Integrals

\[ \int_{1}^{e} \frac{\ln x}{x^2} \, dx \]

you might try a comparison, keep the \( \frac{1}{x^2} \) away the \( \ln x \)

Then you'd have \( \int_{1}^{e} \frac{1}{x^2} \, dx \) which converges.

So you want the convergence part of the comparison test.

\[ \frac{\ln x}{x^2} \leq \frac{1}{x^2} \] \( or \) \( x \leq e \).

So that won't work.

This is a case of integration by parts.

Let \( u = \ln x \) if \( du = \frac{1}{x} \) then \( v = -\frac{1}{x} \).

If not it's a part-

\[ \int_{1}^{e} \frac{\ln x}{x^2} \, dx = -\ln x \cdot \frac{1}{x} - \int \frac{1}{x^2} \, dx \]

\[ \int_{1}^{e} \frac{\ln x}{x^2} \, dx = -\ln x + \frac{1}{x} \]

\[ \int_{1}^{e} \frac{\ln x}{x^2} \, dx = -\ln x + \frac{1}{x} + C \]

So \( \int_{1}^{e} \frac{\ln x + 1}{x} \, dx = -\ln x + \frac{1}{x} + C \)

And now \( \lim_{b \to \infty} \int_{1}^{b} \frac{\ln x}{x^2} \, dx = -\lim_{b \to \infty} \ln b + \frac{1}{b} \leq \frac{1}{2} \)

\[ \lim_{b \to \infty} \frac{1}{b} + \frac{1}{2} = \frac{1}{2} \leq 0. \] So converges.
\[
\int \frac{x}{x^2 + \ln x} \, dx \quad \text{keep the big terms} - \frac{1}{x^2} \text{ is bigger than } \ln x
\]

so this is like \( \int \frac{x}{x^2} \, dx = \int \frac{1}{x} \, dx \) by 

\[ p \text{-test, } p=1 \leq 1. \]

So \( \lim \text{comp tells} \lim_{x \to \infty} \frac{x}{x^2 + \ln x} \)

\[
= \lim_{x \to \infty} \frac{1}{x^2 + \ln x} \cdot \frac{x}{x} = \frac{1}{x + \frac{\ln x}{x}} 
\]

\[
= \lim_{x \to \infty} \frac{1}{x + \frac{\ln x}{x}} - \lim_{x \to \infty} \frac{1}{1 + \frac{\ln x}{x}} = \frac{1}{1} = 1
\]

\[
1 \leq x \leq \infty \text{ so } \int_{1}^{\infty} \frac{x}{x^2 + \ln x} \, dx \text{ by }
\]

\( \lim \text{ comp with } \int_{1}^{\infty} \frac{x}{x^2} \, dx \).

By the way, basic comp wouldn't work - I'd need the divergence part. I'd need \( \frac{x}{x^2 + \ln x} \leq \frac{x}{x^2} \)

or \( \frac{1}{x^2 + \ln x} \leq \frac{1}{x^2} \) or \( x^2 > x^2 + \ln x \) or \( x > \ln x \)

or \( e^{x/2} \ln x \) or \( e^{x/2} \ln x \) or \( 1 \leq x \leq 5 \) or \( x \leq 5 \).

So I needed \( \lim \text{comp.} \)
now I'm supposed to do

\[ \int_{1}^{2} \frac{1}{x^2 + \ln x} \, dx \]

I could do this again,

but what I want to show is that here basic
conv is faster.

OK: 0 think it's like \( \int_{1}^{2} \frac{1}{x^2} \, dx \) which conv
by \( p \)-test \( p = 2 > 1 \). So I need the conv
part of the basic conv test:

\[
\frac{1}{x^2 + \ln x} \leq \frac{1}{x^2} \ \ \ \ \ \ \ x^2 \leq x^2 + \ln x
\]

\[ 0 \leq \ln x \leq \ln x \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ 1 \leq x. \]

But since I have \( \int_{1}^{2} \), \( \forall \ 1 \leq x \) is true!!